

Quantum Stochastic Calculus on Full Fock Modules¹

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stochastic integrals are limits of Riemann–Stieltjes sums. After having established existence and uniqueness of solutions of a large class of quantum stochastic differential equations, we find necessary and sufficient conditions for unitarity of a subclass of solutions. As an application we find dilations of a conservative CP-semigroup (quantum dynamical semigroup) on \mathbf{B} with arbitrary bounded (Christensen–Evans) generator. We point out that in the case $\mathbf{B} = \mathcal{B}(G)$ the calculus may be interpreted as a calculus on the full Fock space tensor initial space G with arbitrary degree of freedom dilating CP-semigroups with arbitrary Lindblad generator. Finally, we show how a calculus on the boolean Fock module reduces to our calculus. As a special case this includes a calculus on the boolean Fock space.

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1. INTRODUCTION

The beginning of quantum stochastic calculus as it is used today is probably the calculus on the *symmetric Fock space* $\Gamma(L^2(\mathbb{R}^+, H))$ by Hudson and Pathasarathy [HP84]. (Compare, however, also the works [AH84] devoted to a calculus on the Fermi Fock space and [BSW82] devoted to the Clifford integral.) The *free* calculus on the *full Fock space* $\mathcal{F}(L^2(\mathbb{R}^+))$ was introduced by Kümmerer and Speicher [KS92, Spe91]. Shortly later Fagnola [Fag91] showed that free calculus fits after very slight modifications into the *representation free* calculus in Accardi, Fagnola, and Quaegebeur [AFQ92].

One of the main goals of quantum stochastic calculus is to find dilations of conservative CP-semigroups (i.e., semigroups of unital completely positive mappings on a unital C^* -algebra B). In usual approaches the *initial algebra* B is taken into account by considering the tensor product of the Fock space by an *initial space* G on which B is represented. In the calculi in [HP84, KS92, Spe91] the dilation problem has been solved for special CP-semigroups on $\mathcal{B}(G)$, namely, those with (bounded) Lindblad generator [Lin76] of *one degree of freedom* (i.e., the one-particle sector is $L^2(\mathbb{R}^+)$ and in each of the possibly infinite sums of Eq. (13.1) only one summand remains). A general Lindblad generator (for separable G) requires a calculus with arbitrary degree of freedom (with one-particle sector $L^2(\mathbb{R}^+, H)$). For the symmetric calculus this problem was solved in [MS90] where infinite sums of integrators appear. A similar calculus on the full Fock space was treated in [FM92]. However, this calculus is only for one-sided integrals and the conservation integral is only mentioned.

Here we develop a free calculus on the *full Fock module*. Already in the case of Lindblad generators (i.e., CP-semigroups on $\mathcal{B}(G)$) it has enormous advantages using Hilbert modules just as a language. The initial space disappears. Instead, we consider two-sided Hilbert modules over $\mathcal{B}(G)$. The infinite sums of integrators are replaced by a finite sum (just one summand for creation, annihilation, conservation and for time integral). We explain this in Section 13.

However, a calculus on Fock modules does more. It allows us to find dilations for (bounded) generators of CP-semigroups on arbitrary C^* -algebras B whose form was found by Christenson and Evans [CE79]. Recently, Goswami and Sinha [GS99] introduced a calculus on a *symmetric Fock module* [Ske98a] and used it to solve the dilation problem for Christensen–Evans generators.

As usual, the one-particle sector is obtained by a GNS-construction from the generator and, therefore, it is a Hilbert B – B -module. One problem which had to be faced in [GS99] is that, as pointed out in [Ske98a], the symmetric Fock module over an arbitrary Hilbert B – B -module does not

exist without additional assumptions. One sufficient assumption is that the Hilbert module is *centered* (i.e., it is generated by those elements which commute with B). In [BS99] we point out that Hilbert $\mathcal{B}(G)$ – $\mathcal{B}(G)$ -modules (complete in a certain weak topology) are always centered. And, indeed, in [GS99] it is one of the first steps to embed the the Hilbert B – B -module which arises by GNS-construction into a bigger Hilbert $\mathcal{B}(G)$ – $\mathcal{B}(G)$ -module. On the contrary, a full Fock module can be constructed over arbitrary one particle sectors. Therefore, in our case we do not have to enlarge the one-particle sector.

A first attempt for a calculus on a full Fock module was made in Lu [Lu94] where the calculus lives on the Fock module $\mathcal{F}(L^2(\mathbb{R}^+, A))$ (instead of $\mathcal{F}(L^2(\mathbb{R}^+, E))$). As A is the simplest A – A -module possible, the module structure of the one-particle sector is very simple. In fact, the calculus is *isomorphic* to the calculus on $G \otimes \mathcal{F}(L^2(\mathbb{R}^+))$ in [KS92] where A is represented on G . However, the algebra $A = \mathcal{B}^a(\mathcal{F}(E))$ is very big and contains the original algebra B only as a, usually, very small subalgebra. We also mention the abstract calculus by Hellmich, Köstler, and Kümmerer as outlined in [HKK98] where a one-to-one correspondence between additive and multiplicative adapted cocycles with respect to an abstract white noise is established. These results are, however, restricted to the set-up of von Neumann algebras with faithful normal (invariant) states.

Our approach to calculus is inspired very much by [KS92] and we borrowed also some essential ideas from [Spe91] as far as conservation integrals are concerned. Reference [KS92] develops stochastic integration for creation and annihilation processes. All limits there are norm limits. Taking into account also conservation integrals destroys norm convergence. In [Spe91] this problem is solved with the help of a kernel calculus. We follow, however, the ideas in [Ske98b] and use the $*$ -strong topology, dealing always with concrete operators.

The basic idea in [KS92, Spe91] is probably to use the graduation on the Fock space in order to define a new norm. It is this idea which is responsible for the fact that we are in a position to find a calculus of bounded operators. In Section 2 we repeat this idea in a general set-up. Additionally, we proof the necessary generalizations for strong limits.

In Section 3 we define the full Fock module and basic operators on it. In particular, the generalized creators and annihilators as introduced in [Ske98b] simplify notation in the following sections considerably. In Section 4 we define adaptedness, again following [Ske98b]. This notion of adaptedness, is simpler and more general than the original notion in [KS92]. Also here the generalized creators play a crucial role in drawing consequences of adaptedness in a transparent way.

In [KS92] the theory is developed for processes which belong to some L^4 -space. This is in some sense the most general class possible. Here we consider $*$ -strongly continuous processes. This is sufficient, because all

integrals lead to processes belonging to this class. Additionally, our restriction has the advantage that all integrals are limits of Riemann–Stieltjes sums. On the other hand, our theory is dealing with very general integrators (whereas the integrators in [KS92, Spe91] are the simplest possible). In fact, our integrators are so general that the differential equation resolving the dilation problem has not a single coefficient. In Section 5 we introduce both the function spaces from which we take our processes and those from which we take the integrators.

In Section 6 we show existence of integrals for the considered class of processes and integrators. In Section 8 we show that conservation integrals are essentially non-continuous. In Section 9 we establish the Ito formula. As the techniques used here depend highly on the class of processes and integrators, these sections differ considerably from the corresponding sections in [KS92]. In particular, the results in Section 8 are much more involved than the corresponding results in [Spe91].

In Section 7 we show existence and uniqueness of solutions of differential equations. In Section 11 we establish that solutions of particular differential equations, those with “stationary independent increments,” have cocycles as solutions. In Section 10 we state necessary and sufficient conditions for unitarity of the solution and in Section 12 we use the results to solve the dilation problem for a general Christensen–Evans generator. The ideas to all proofs in these sections are taken directly from [KS92, Spe91]. It is noteworthy that, actually, the proofs here, although more general, are formally simpler than the original proofs. (This is due to absence of coefficients in our differential equations.)

We close with some applications. In Section 13 we explain that the calculus on the full Fock space $G \otimes \mathcal{F}(L^2(\mathbb{R}, H))$ ([KS92, Spe91] treated only the case $H = \mathbb{C}$) is contained in our set-up. In Section 14 we show that the calculus on the *boolean Fock module* is included. In particular, we show that the (non-conservative) CP-semigroups T on \mathbf{B} which may be dilated with the help of a boolean calculus are precisely those having the form $T_t(b) = b_t b b_t^*$ where $b_t = e^{tJ}$ ($J \in \mathbf{B}$, $\operatorname{Re} J \leq 0$) is a semigroup of contractions in \mathbf{B} .

Conventions and Notations. In the sequel, in constructs like “quantum stochastic calculus” or “quantum stochastic differential equation,” etc., we leave out the words “quantum stochastic.” By \mathbb{R}^+ and \mathbb{N}_0 we denote the non-negative reals and integers, respectively. $\mathbb{N} = \{1, 2, \dots\}$.

For basics about Hilbert modules we refer the reader to [Pas73, Lan95, Ske97, BS99]. We recall only that for us Hilbert \mathbf{B} -modules are right \mathbf{B} -modules with a \mathbf{B} -valued inner product, right \mathbf{B} -linear in its right variable. Hilbert \mathbf{A} – \mathbf{B} -modules are Hilbert \mathbf{B} -modules where \mathbf{A} acts non-degenerately as a C^* -algebra of right module homomorphisms. In

particular, if A is unital, the unit of A acts as unit. The C^* -algebra of adjointable mappings on a Hilbert module E we denote by $\mathcal{B}^a(E)$. The (interior) tensor product (over B) of the Hilbert A - B -module E and the Hilbert B - C -module F is the Hilbert A - C -module $E \odot F$ with inner product defined by setting $\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$. Recall that by definition Hilbert modules are complete with respect to their norm $\|x\| = \sqrt{\|\langle x, x \rangle\|}$. Otherwise, we speak of pre-Hilbert modules. The strong topology is that of operators on a normed or Banach space. The $*$ -strong topology on an involutive space of operators on a normed or Banach space is the topology generated by the strong topology and by the strong topology for the adjoints. (When restricted to bounded subsets of $\mathcal{B}^a(E)$ this is the strict topology; see [Lan95].)

2. OPERATORS ON GRADED BANACH SPACES

Fock spaces or Fock modules, in the first place, are *graded vector spaces*. More precisely, they contain a dense graded vector subspace, the algebraic direct sum over the n -particle sectors. The \mathbb{Z} -graduation on Fock type spaces gives rise to a natural graduation on the spaces of linear operators among them. Unfortunately, the graduation is not totally compatible with the inner product topology on the Fock spaces. As a consequence, the graded vector subspace in a space of operators on a Fock space is not norm dense. (Of course, it is strongly dense.) The solution of this problem, already found in the first paper [KS92] by Kümmerer and Speicher, was the beginning of free quantum stochastic calculus. The idea is, roughly speaking, to introduce a stronger norm on graded *Banach spaces* which behaves more nicely with respect to bilinear mappings. We follow this basic idea. The benefit of this approach is an operator calculus of bounded operators with strong convergence on the whole domain (even norm convergence when conservation integrals are excluded). The price to be paid is the exclusion of some (even bounded) operators. However, these operators can never appear as solutions of a quantum stochastic differential equation.

Let $(V^{(n)})_{n \in \mathbb{Z}}$ be a family of Banach spaces. By V_1 we denote their ℓ^1 -direct sum. In other words, V_1 is the completion of the algebraic direct sum $V_g := \bigoplus_{n \in \mathbb{Z}} V^{(n)}$ with respect to the norm

$$\|v\|_1 = \sum_{n \in \mathbb{Z}} \|v^{(n)}\|.$$

Clearly, V_1 consists of all families $v = (v^{(n)})$ ($v^{(n)} \in V^{(n)}$) for which $\|v\|_1 < \infty$. Taking into account that V_g is a (\mathbb{Z}) -graded vector space, we call the

elements of $V^{(n)}$ homogeneous of degree n . (We do not exclude $0 \in V^{(n)}$ ($n \in \mathbb{Z}$).)

The following lemma differs from a result in [KS92] just by a slightly more general formulation. Together with Lemma 2.2 it shows us that all limits of bilinear mappings in the subsequent sections have to be computed only when evaluated at homogeneous elements.

2.1 LEMMA. *Let $(V^{(n)})_{n \in \mathbb{Z}}$, $(C^{(n)})_{n \in \mathbb{Z}}$, and $(D^{(n)})_{n \in \mathbb{Z}}$ be families of Banach spaces. Suppose that $j: C_g \times D_g \rightarrow V_g$ is an even bilinear mapping (i.e., $j(C^{(n)}, D^{(m)}) \subset V^{(n+m)}$ for all $n, m \in \mathbb{Z}$), and that $M > 0$ is a constant such that*

$$\|j(c, d)\|_1 \leq M \|c\|_1 \|d\|_1 \quad (2.1)$$

for all homogeneous $c \in C_1$, $d \in D_1$. Then j extends to a (unique) bilinear mapping $C_1 \times D_1 \rightarrow V_1$, also denoted by j , such that (2.1) is fulfilled for all $c \in C_1$, $d \in D_1$. (In other words, j is bounded.)

Proof. We show that (2.1) extends to arbitrary $c \in C_g$, $d \in D_g$. (Of course, such a mapping j extends by means of continuity to a unique bilinear mapping on $C_1 \times D_1$ also fulfilling (2.1).) Indeed,

$$\begin{aligned} \|j(c, d)\|_1 &= \left\| \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} j(c^{(m)}, d^{(n-m)}) \right\|_1 \\ &\leq M \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \|c^{(m)}\| \|d^{(n-m)}\| = M \|c\|_1 \|d\|_1. \quad \blacksquare \end{aligned}$$

Let V be a Banach space with a family $(V^{(n)})_{n \in \mathbb{Z}}$ of mutually linearly independent Banach subspaces. Then $\|v\| \leq \|v\|_1$ for all $v \in V_g$. In other words, we may identify V_1 as a subspace of V and the inequality $\|v\| \leq \|v\|_1$ extends to all elements $v \in V_1$.

Now consider the Banach space $\mathcal{B}(V)$ of bounded (linear) operators on V . For each $n \in \mathbb{Z}$ we denote by

$$\mathcal{B}(V)^{(n)} = \{a \in \mathcal{B}(V) : aV^{(m)} \subset V^{(n+m)}\}$$

the Banach space of all bounded operators on V which are homogeneous of degree n . As later on $V^{(n)}$ will typically be the n -particle sector of a Fock module, we call $\mathcal{B}(V)^{(n)}$ the space of operators with *offset n in the number of particles*.

2.2. LEMMA. *Let $(j_\lambda)_{\lambda \in A}$ be a net of even bilinear mappings $j_\lambda: C_1 \times D_1 \rightarrow \mathcal{B}(V)_1$ (indexed by some directed set A) all fulfilling (2.1) with a constant $M > 0$ which is independent of λ . Furthermore, suppose that for homogeneous*

$c \in C_1$, $d \in D_1$ the net $j_\lambda(c, d)$ converges strongly in $\mathcal{B}(V)$ (of course, to a homogeneous element in $\mathcal{B}(V)$). Then the mapping $(c, d) \mapsto \lim_\lambda j_\lambda(c, d)$ on $C_g \times D_g$ fulfills (2.1) and, therefore, extends by Lemma 2.1 to a mapping $j: C_1 \times D_1 \rightarrow \mathcal{B}(V)_1$. Moreover, $j_\lambda(c, d)$ converges strongly in $\mathcal{B}(V)$ to $j(c, d)$ for all $c \in C_1$, $d \in D_1$.

Proof. Let $c \in C_1$, $d \in D_1$, $v \neq 0$ in V and $\varepsilon > 0$. We may choose $c_g \in C_g$, $d_g \in D_g$ such that

$$\|j_\lambda(c, d) - j_\lambda(c_g, d_g)\|_1 < \frac{\varepsilon}{3 \|v\|} \quad \text{and} \quad \|j(c, d) - j(c_g, d_g)\|_1 < \frac{\varepsilon}{3 \|v\|}$$

for all $\lambda \in A$. Furthermore, choose $\lambda_0 \in A$ such that

$$\|j(c_g, d_g) v - j_{\lambda_0}(c_g, d_g) v\| < \frac{\varepsilon}{3}$$

for all $\lambda \geq \lambda_0$. ■

2.3. Remark. Lemmata 2.1 and 2.2 have obvious generalizations to multi-linear even mappings. This can be shown by direct generalization of the above proofs. A less direct but more elegant method makes use of the *projective tensor product* of Banach spaces. Almost everything that can be said about the projective norm on the algebraic tensor product of two Banach spaces is collected in the remarkable Proposition T.3.6 in [WO93, Appendix T]. Recall that the projective norm is that unique cross norm which carries over the universal property of the tensor product to the context of Banach spaces. In other words, there is a one-to-one correspondence between bounded bilinear mappings on two Banach spaces and bounded linear mappings on their projective tensor product. Moreover, the projective norm majorizes any other subcross norm on the algebraic tensor product. The tensor product of two graded vector spaces carries a natural graduation. Of course, the $\|\bullet\|_1$ -norm constructed on the projective tensor product from this graduation majorizes the projective norm. On the other hand, one easily verifies that it is subcross. Consequently, it must coincide with the projective norm. Now the multi-linear analogues of the lemmata follow easily by induction, when translated into statements on linear mappings on multiple projective tensor products.

Clearly, Lemmata 2.1 and 2.2 (and their multi-linear extensions) remain also true in the case, when j is *homogeneous of degree ℓ* , i.e., when $j(C^{(n)}, D^{(m)}) \subset V^{(n+m+\ell)}$ ($n, m \in \mathbb{Z}$).

2.4. COROLLARY. *The convergence in Lemma 2.2 is also strongly in $\mathcal{B}(V_1)$.*

Proof. $(c, d, v) \mapsto j_\lambda(c, d)v$ is a 3-linear mapping on $C_1 \times D_1 \times V_1$. Therefore, we may also replace $v \in V_1$ by an element $v_g \in V_g$ which is close to v in $\|\bullet\|_1$. ■

3. OPERATORS ON FULL FOCK MODULE

In this section we introduce the Fock module over a two-sided Hilbert module. The definition of a Fock module is due to Pimsner [Pim97] and Speicher [Spe98]. The C^* -algebra generated by the creators is analyzed in [Pim97]. The conservator is introduced in [Spe98]. The generalized creators are introduced in [Ske98b]. They also appear naturally, if we want to explain why Arveson's spectral algebra [Arv90] is the continuous time analogue of the Cuntz algebra [Cun77]. Here we need them to describe most conveniently the algebraic consequences of adaptedness in Section 4.

3.1. DEFINITION. Let \mathbf{B} be a unital C^* -algebra and let E be a Hilbert \mathbf{B} - \mathbf{B} -module. Then the *full Fock module over E* is the Hilbert \mathbf{B} - \mathbf{B} -module

$$\mathcal{F}(E) = \bigoplus_{n=0}^{\infty} E^{\odot n},$$

where $E^{\odot 0} = \mathbf{B}$ and $\omega = \mathbf{1} \in E^{\odot 0}$ is the *vacuum*. The *vacuum conditional expectation* $\mathbb{E}_0: \mathcal{B}^a(\mathcal{F}(E)) \rightarrow \mathbf{B}$ is defined, by setting $\mathbb{E}_0(a) = \langle \omega, a\omega \rangle$.

We define the *homogeneous* subspaces $E^{(n)} = E^{\odot n}$ ($n \in \mathbb{N}_0$) and $E^{(n)} = \{0\}$ ($n < 0$). We denote by $\mathcal{F}_g(E)$ and $\mathcal{F}_1(E)$ the algebraic direct sum and the ℓ^1 -direct sum, respectively, over all $E^{(n)}$. In other words, $\mathcal{F}_1(E)$ consists of all families $(x^{(n)})_{n \in \mathbb{Z}}$ ($x^{(n)} \in E^{(n)}$) for which $\|x\|_1 = \sum_{n \in \mathbb{Z}} \|x^{(n)}\| < \infty$. Since $\|x\| \leq \|x\|_1$, we have $\mathcal{F}_g(E) \subset \mathcal{F}_1(E) \subset \mathcal{F}(E)$.

3.2. DEFINITION. For $n \in \mathbb{Z}$ we denote by $\mathcal{B}^{(n)} \subset \mathcal{B}^a(\mathcal{F}(E))$ the space consisting of all operators with *offset n in the number of particles*, i.e., $a^{(n)} \in \mathcal{B}^{(n)}$, if $a^{(n)}(E^{\odot m}) \subset E^{\odot(m+n)}$. Also $\mathcal{B}^a(\mathcal{F}(E))$ has a natural graded vector subspace \mathcal{B}_g with $\mathcal{B}^{(n)}$ ($n \in \mathbb{Z}$) being the *homogeneous* subspaces. Any $a \in \mathcal{B}^a(\mathcal{F}(E))$ allows a $*$ -strong decomposition into $a = \sum_{n \in \mathbb{Z}} a^{(n)}$ with $a^{(n)} \in \mathcal{B}^{(n)}$. We define the Banach space \mathcal{B}_1 as the space consisting of all $a \in \mathcal{B}^a(\mathcal{F}(E))$ for which $\|a\|_1 = \sum_{n \in \mathbb{Z}} \|a^{(n)}\| < \infty$. Again, we have $\|a\| \leq \|a\|_1$, so that $\mathcal{B}_g \subset \mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E))$.

Obviously, $\mathcal{B}^{(n)}\mathcal{B}^{(m)} \subset \mathcal{B}^{(n+m)}$ so that the multiplication on \mathcal{B}_g is an even mapping. Notice also that $\mathcal{B}^{(n)*} \subset \mathcal{B}^{(-n)}$. By Lemma 2.1, \mathcal{B}_1 is a Banach $*$ -algebra. $\mathcal{B}^a(\mathcal{F}(E))$ is $*$ -strongly complete and, therefore, so is the closed subspace $\mathcal{B}^{(n)}$.

3.3. DEFINITION. Let $x \in E$. The *creation operator* (or *creator*) $\ell^*(x)$ on $\mathcal{F}(E)$ is defined by setting

$$\ell^*(x) x_n \odot \cdots \odot x_1 = x \odot x_n \odot \cdots \odot x_1$$

for $n \geq 1$ and $\ell^*(x) \omega = x$. The *annihilation operator* (or *annihilator*) is the adjoint operator, i.e.,

$$\ell(x) x_n \odot \cdots \odot x_1 = \langle x, x_n \rangle x_{n-1} \odot \cdots \odot x_1$$

for $n \geq 1$ and 0 otherwise. Let $T \in \mathcal{B}^a(E)$. The *conservation operator* (or *conservator*) $p(T)$ on $\mathcal{F}(E)$ is defined by setting

$$p(T) x_n \odot \cdots \odot x_1 = (Tx_n) \odot x_{n-1} \odot \cdots \odot x_1$$

for $n \geq 1$ and 0 otherwise.

For any \mathbf{B} - \mathbf{B} -linear mapping $T \in \mathcal{B}^a(E)$ we may define its *second quantization*

$$\mathcal{F}(T) = \bigoplus_{n \in \mathbb{N}_0} T^{\odot n} \in \mathcal{B}^a(\mathcal{F}(E)) \quad (T^{\odot 0} = \text{id}).$$

3.4. PROPOSITION. We have $\ell^*(x) \in \mathcal{B}^{(1)}$, $p(T) \in \mathcal{B}^{(0)}$, and $\ell(x) \in \mathcal{B}^{(-1)}$. The mappings $x \mapsto \ell^*(x)$ and $T \mapsto p(T)$ depend \mathbf{B} - \mathbf{B} -linearly on their arguments. The mapping $x \mapsto \ell(x)$ depends \mathbf{B} - \mathbf{B} -anti-linearly on its argument. We have $\|\ell^*(x)\| = \|\ell(x)\| = \|x\|$ and $\|p(T)\| = \|T\|$.

We have

$$p(TT') = p(T) p(T') \quad \text{and} \quad p(T^*) = p(T)^*$$

so that $T \mapsto p(T)$ defines an injective homomorphism of C^* -algebras. Finally, we have the relations

$$p(T) \ell^*(x) = \ell^*(Tx), \quad \ell(x) p(T) = \ell(T^*x), \quad \ell(x) \ell^*(x') = \langle x, x' \rangle.$$

Proof. The other statements being obvious, we only show $\|p(T)\| = \|T\|$ and postpone $\|\ell^*(x)\| = \|x\|$ to the more general statement in Proposition 3.7. We have $p(T) = 0 \oplus T \odot \text{id}$ on $\mathcal{F}(E) = \mathbf{B}\omega \oplus E \odot \mathcal{F}(E)$. Therefore, $\|p(T)\| \leq \|T\|$. On the other hand, $p(T) \upharpoonright E^{\odot 1} = T$ so that $\|p(T)\|$ certainly is not smaller than $\|T\|$. ■

3.5. DEFINITION. Let $X \in \mathcal{F}(E)$. By the *generalized creator* $\hat{\ell}^*(X)$ we mean the operator on $\mathcal{F}(E)$ defined by setting

$$\hat{\ell}^*(X) Y = X \odot Y$$

for $Y \in E^{\odot n}$, where we identify $\mathcal{F}(E) \odot E^{\odot n}$ as a subset of $\mathcal{F}(E)$ in an obvious way (cf. the proof of Proposition 4.1). The *generalized annihilator* $\hat{\ell}(X)$ is the adjoint of $\hat{\ell}^*(X)$.

3.6. Remark. For $Y \in E^{\odot n}$ we easily find $\|\hat{\ell}^*(X) Y\| \leq \|X\| \|Y\|$. However, it is not difficult to see that $\hat{\ell}^*(X)$ is not necessarily a bounded operator on $\mathcal{F}(E)$.

For $X \in E^{\odot n}$ we find

$$\hat{\ell}(X) x_{n+m} \odot \cdots \odot x_1 = \langle X, x_{n+m} \odot \cdots \odot x_{m+1} \rangle x_m \odot \cdots \odot x_1$$

and $\hat{\ell}(X) E^{\odot m} = \{0\}$, if $m < n$.

3.7. PROPOSITION. *Let $X \in E^{\odot n}$. Then $\hat{\ell}^*(X) \in \mathcal{B}^{(n)}$ and $\hat{\ell}(X) \in \mathcal{B}^{(-n)}$. We have $\|\hat{\ell}^*(X)\| = \|\hat{\ell}(X)\| = \|X\|$. For $T \in \mathcal{B}^a(E)$ we have*

$$p(T) \hat{\ell}^*(X) = \hat{\ell}^*(p(T) X),$$

where we consider X also as an element of $\mathcal{F}(E)$. Moreover, for $Y \in E^{\odot m}$ we have

$$\hat{\ell}(X) \hat{\ell}^*(Y) = \hat{\ell}^*(\hat{\ell}(X) Y) \quad \text{or} \quad \hat{\ell}(X) \hat{\ell}^*(Y) = \hat{\ell}(\hat{\ell}(Y) X)$$

depending on whether $n < m$ or $n > m$. For $n = m$ we have

$$\hat{\ell}(X) \hat{\ell}^*(Y) = \langle X, Y \rangle. \quad (3.1)$$

Proof. We only show $\|\hat{\ell}^*(X)\| = \|X\|$. This follows easily from (3.1), because for $Y \in \mathcal{F}(E)$ we have $\|\hat{\ell}^*(X) Y\|^2 = \|\langle Y, \langle X, X \rangle Y \rangle\|^2 \leq \|\langle X, X \rangle\| \|\langle Y, Y \rangle\| = \|X\|^2 \|Y\|^2$. ■

3.8. COROLLARY. *For $X \in \mathcal{F}_1(E)$ we have $\|\hat{\ell}^*(X)\|_1 = \|\hat{\ell}(X)\|_1 = \|X\|_1$. In particular, we find for $a \in \mathcal{B}_1$ that $\|\hat{\ell}^*(a\omega)\| \leq \|\hat{\ell}^*(a\omega)\|_1 = \|a\omega\|_1 \leq \|a\|_1$ so that $\hat{\ell}^*(a\omega)$ is a well-defined element of $\mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E))$.*

3.9. COROLLARY. *Let $a_t \in \mathcal{B}_1$ such that $t \mapsto a_t$ is strongly continuous in $\mathcal{B}^a(\mathcal{F}(E))$. Then both mappings $t \mapsto \hat{\ell}^*(a_t\omega)$ and $t \mapsto \hat{\ell}(a_t\omega)$ are $\|\bullet\|_1$ -continuous.*

Proof. By an argument very similar to the proof of Lemma 2.2, we see that $t \mapsto a_t$ is strongly continuous also in \mathcal{B}_1 . Now the statement follows easily from Corollary 3.8. ■

3.10. *Remark.* $\mathcal{F}_1(E)$, equipped with the multiplication obtained from the multiplication of the \mathbf{B} -tensor algebra $\mathcal{F}_g(E)$ (see [Ske98a]) and continuous extension in $\|\bullet\|_1$, is a Banach algebra. Corollary 3.8 tells us that $\hat{\ell}^*$ and $\hat{\ell}$ are an isometric homomorphism and an isometric (anti-linear) anti-homomorphism, respectively, into \mathcal{B}_1 .

4. ADAPTEDNESS

4.1. **PROPOSITION.** *Let E, F be Hilbert \mathbf{B} - \mathbf{B} -modules. Then*

$$\mathcal{F}(E \oplus F) \cong \mathcal{F}(E) \odot (\mathbf{B}\omega \oplus F \odot \mathcal{F}(E \oplus F)) \quad (4.1)$$

in a canonical way.

Proof. Let $n, m \geq 0$, $x_i \in E$ ($i = 1, \dots, n$), $y \in F$, $z_j \in E \oplus F$ ($j = 1, \dots, m$). We easily check that the mapping, sending $(x_n \odot \dots \odot x_1) \odot (y \odot z_m \odot \dots \odot z_1)$ on the right-hand side to $x_n \odot \dots \odot x_1 \odot y \odot z_m \odot \dots \odot z_1$ on the left-hand side (and sending $(x_n \odot \dots \odot x_1) \odot \omega$ to $x_n \odot \dots \odot x_1$, and $\omega \odot (y \odot z_m \odot \dots \odot z_1)$ to $y \odot z_m \odot \dots \odot z_1$), and, of course, sending $\omega \odot \omega$ to ω) extends as an isometry onto $\mathcal{F}(E \oplus F)$. ■

This factorization was found first for Fock spaces in [Fow95]. We used it, independently, in [Ske98b] in the context of quantum stochastic calculus, in order to describe adapted operators.

4.2. **DEFINITION.** An operator a in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ is called *adapted to E* , if there is an operator $a_E \in \mathcal{B}^a(\mathcal{F}(E))$ such that $a = (a_E \odot \text{id})$ in the decomposition according to (4.1). Applying $a_E \odot \text{id}$ to vectors of the form $x \odot \omega$, we see that a_E is unique.

4.3. *Observation.* By definition, the set of all operators adapted to E is precisely

$$\mathcal{B}^a(\mathcal{F}(E)) \odot \text{id} \cong \mathcal{B}^a(\mathcal{F}(E)).$$

(This identification is an isomorphism of C^* -algebras. The $*$ -strong topology is, in general, not preserved.) The identification is canonical in the sense that it identifies creators to the same element $x \in E$. Indeed, the creator $\ell^*(x) \in \mathcal{B}^a(\mathcal{F}(E))$ ($x \in E$) embedded via $(\ell^*(x) \odot \text{id})$ into $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ coincides with the creator $\ell^*(x) \in \mathcal{B}^a(\mathcal{F}(E \oplus F))$ where now x is considered as an element of $E \oplus F$. The $*$ -algebra generated by all creators to elements $x \in E$ is $*$ -strongly dense in $\mathcal{B}^a(\mathcal{F}(E))$, and we may identify the $*$ -subalgebra of $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ consisting of all operators adapted to E

with the $*$ -strong closure in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ of the $*$ -algebra generated by all creators on $\mathcal{F}(E \oplus F)$ to elements in $E \subset E \oplus F$.

Under the above isomorphism also the Banach $*$ -algebra $\mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E))$ coincides (isometrically in $\|\bullet\|_1$) with the Banach $*$ -algebra of all elements in $\mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E \oplus F))$ which are adapted to E .

4.4. COROLLARY. *Let $x \in E$, $T \in \mathcal{B}^a(E)$ and $X \in \mathcal{F}_1(E)$. Then $\ell^*(x)$, $\ell(x)$, $p(T)$, $\hat{\ell}^*(X)$ and $\hat{\ell}(X)$ are adapted to E . Also the identity is adapted. Moreover, $\hat{\ell}^*(X) \in \mathcal{B}_1$ is adapted to E , if and only if $X \in \mathcal{F}_1(E)$.*

4.5. LEMMA. *Let $a \in \mathcal{B}_1$ be adapted to E and T in $\mathcal{B}^a(F)$. Then*

$$ap(T) = \hat{\ell}^*(a\omega) p(T) \quad (4.2a)$$

and

$$p(T) a = p(T) \hat{\ell}(a^*\omega). \quad (4.2b)$$

Proof. As (4.2b) is more or less the adjoint of (4.2a), it is sufficient only to prove (4.2a).

Equation (4.2a) follows from the observation that the range of $p(T)$ is contained in $(F \odot \mathcal{F}(E \oplus F))$ and from $a_E \omega = a\omega$ in the identification $\mathcal{F}(E) \subset \mathcal{F}(E \oplus F)$. ■

4.6. COROLLARY. *Let $a, b \in \mathcal{B}_1$ both be adapted to E and let T, T' be in $\mathcal{B}^a(F)$. Then*

$$p(T) abp(T') = p(T \mathbb{E}_0(ab) T').$$

Proof. By Corollary 3.8 we may assume that $a \in \mathcal{B}^{(n)}$ and $b \in \mathcal{B}^{(m)}$. First, suppose $-n \neq m$. Then $\mathbb{E}_0(ab) = 0$. Without loss of generality we may assume $-n < m$. From Proposition 3.7 and Lemma 4.5 we find

$$p(T) abp(T') = \hat{\ell}^*(p(T) \hat{\ell}(a^*\omega) b\omega) p(T') = 0,$$

because $\hat{\ell}(a^*\omega) b\omega$ is an element of $E^{\odot(n+m)}$ and T vanishes on E . If $n = m$, we find $p(T) abp(T') = p(T) \hat{\ell}(a^*\omega) \hat{\ell}^*(b\omega) p(T') = p(T) \mathbb{E}_0(ab) p(T')$. Therefore, in both cases we obtain our claimed result. ■

4.7. COROLLARY. *Suppose $a \in \mathcal{B}^{(0)}$ is adapted to E and $T \in \mathcal{B}^a(F)$. Then*

$$ap(T) = \mathbb{E}_0(a) p(T).$$

5. BILINEAR MAPPINGS ON SPACES OF BANACH SPACE VALUED FUNCTIONS

In these notes we are dealing with integrals $\int_{\tau}^{\mathcal{T}} F_t dI_t G_t$ where the integrands F and G are processes of operators on a Fock module and $dI_t = I_{t+dt} - I_t$ are differentials constructed from certain basic integrator processes I . It is typical for free calculus that the integrands do not commute with the differentials. Therefore, we have to work with two integrands F and G . In other words, we investigate bilinear mappings $(F, G) \mapsto \int_{\tau}^{\mathcal{T}} F_t dI_t G_t$.

The starting point of any calculus is to define an integral for *step functions* as Riemann–Stieltjes sum, and then to extend it to a larger class of processes by continuity in a suitable topology. In [KS92] this is done by completion in some L^2 -norm. This works, however, only for creation and annihilation integrals. In [Spe91] a conservation integral is included, making use of a kernel calculus. The price to be paid is that it is difficult to see the concrete operator which is determined by a kernel, explicitly. (It is also hard to see which class of processes, actually, is dealt with.) In [Ske98b] we modified the concrete operator approach from [KS92], by using a $*$ -strong version of the topology. Now also the conservation integral is included, avoiding a kernel approach. Additionally, we decided not to complete in a $*$ -strong version of an L^2 -norm topology, but in a $*$ -strong version of supremum norm topology. Consequently, the calculus in [Ske98b] is dealing with $*$ -strongly continuous processes. One benefit is that all integrals exist as limits of Riemann–Stieltjes sums. In the following section it turns out that existence of all integrals in these notes is covered either by Proposition 5.3 or by Proposition 5.7.

We follow the same approach here (but also an L^2 -version can be done). Since we are interested mainly in adapted processes, the steps in a Riemann–Stieltjes sum take their value from the left border. Consequently, we do not consider all step functions, but only those where indicator functions to left closed and right open intervals are involved. The limits of such functions are precisely the (strongly) right continuous functions with left (strong) limit in each point, the so-called (*strong*) *càdlàg* functions (*continue à droite, limitée à gauche*).

5.1. DEFINITION. Let $K = [\tau, \mathcal{T}]$ ($\tau < \mathcal{T}$) be a compact interval. By

$$\mathcal{P}_K = \{P = (t_0, \dots, t_N) : \tau = t_0 < \dots < t_N = \mathcal{T} (N \in \mathbb{N})\}$$

we denote the *net of partitions of the interval* $[\tau, \mathcal{T}]$ directed increasingly by refinement. The norm of a partition P is $\|P\| = \max_{1 \leq k \leq N} (t_k - t_{k-1})$. Recall that \mathcal{P}_K is a lattice with minimal element. (The unique minimum and

maximum of two element $P, P' \in \mathcal{P}_K$ are given by the “intersection” and the “union” of P and P' . The minimal partition is (τ, \mathcal{T}) .

Let V be a Banach space. For any function $F: K \rightarrow V$ we set $\|F\|^K = \sup_{t \in [\tau, \mathcal{T}]} \|F_t\| \leq \infty$. We say F is *bounded*, if its *norm* $\|F\|^K$ is finite. Let $P \in \mathcal{P}_K$. We set

$$F_t^P = \sum_{i=1}^n F_{t_{k-1}} \chi_{[t_{k-1}, t_k)}(t).$$

By $\mathfrak{S}(K, V)$ we denote the space of *right continuous V -valued step functions on $[\tau, \mathcal{T}]$* . In other words, for each $F \in \mathfrak{S}(K, V)$ there exists a partition $P \in \mathcal{P}_K$ such that $F = F^P$ (and, of course, $F = F^{P'}$ for all $P' \in \mathcal{P}_K$ with $P' \geq P$).

By $\mathcal{C}(K, V)$, and $\mathfrak{R}(K, V)$ we denote the space of *continuous V -valued functions*, and the space of *bounded right continuous V -valued functions with left limit* (or for short, *càdlàg functions*), respectively, on $[\tau, \mathcal{T}]$. The *uniform topology* on $\mathfrak{R}(K, V)$ is just the norm topology.

Let $\mathcal{B} \subset \mathcal{B}(V)$ be a subalgebra of the algebra of bounded operators on V . By $\mathcal{C}^s(K, \mathcal{B})$, and $\mathfrak{R}^s(K, \mathcal{B})$ we denote the space of *strongly continuous \mathcal{B} -valued-functions*, and the space of *bounded strongly right continuous \mathcal{B} -valued functions with strong left limit* (or for short, *strong càdlàg functions*), respectively, on $[\tau, \mathcal{T}]$. On $\mathfrak{R}^s(K, \mathcal{B})$ we define the *strong topology* as the locally convex Hausdorff topology generated by the family $\|\bullet\|_v^K = \|\bullet v\|^K$ ($v \in V$) of seminorms.

5.2. PROPOSITION. (1) $\mathfrak{R}(K, V)$ is a Banach space with respect to the norm $\|\bullet\|^K$, and $\mathcal{C}(K, V)$ is a Banach subspace of $\mathfrak{R}(K, V)$. For all $F \in \mathfrak{R}(K, V)$ we have $F^P \rightarrow F$ in the uniform topology. In other words, the step functions $\mathfrak{S}(K, V)$ form a dense subset of $\mathfrak{R}(K, V)$. Moreover, for each $F \in \mathcal{C}(K, V)$, and each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|F - F^P\|^{K'} < \varepsilon$$

for all compact intervals $K' \subset K$ and all $P \in \mathcal{P}_K$ with $\|P\| < \delta$. We say the continuous functions can be approximated by step functions equiuniformly.

(2) Also $\mathfrak{R}^s(K, \mathcal{B})$ is a Banach space with respect to the norm $\|\bullet\|^K$, and $\mathcal{C}^s(K, \mathcal{B})$ is a Banach subspace of $\mathfrak{R}^s(K, \mathcal{B})$. Moreover, $\mathcal{C}^s(K, \mathcal{B})$ is a strongly closed subset of $\mathfrak{R}^s(K, \mathcal{B})$, and each strongly closed subset of $\mathfrak{R}^s(K, \mathcal{B})$ is also norm closed. For all $F \in \mathfrak{R}^s(K, \mathcal{B})$ we have $F^P \rightarrow F$ in the strong topology. In other words, the step functions $\mathfrak{S}(K, V)$ form a strongly

dense subset of $\mathfrak{R}^s(K, \mathcal{B})$. Moreover, for each $F \in \mathcal{C}^s(K, \mathcal{B})$, each $v \in V$, and each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|F - F^P\|_v^{K'} < \varepsilon$$

for all compact intervals $K' \subset K$ and all $P \in \mathcal{P}_K$ with $\|P\| < \delta$. We say the strongly continuous functions can be approximated by step functions equistrongly.

Proof. It is well known that $\mathcal{C}(K, V)$ is a Banach space. Small modifications of the well-known argument show that also $\mathfrak{R}(K, V)$ is a Banach space. The corresponding statements for $\mathfrak{R}^s(K, \mathcal{B})$ and $\mathcal{C}^s(K, \mathcal{B})$ follow by an application of the *principle of uniform boundedness* and from the observation that the strong topology is weaker than the norm topology.

Density of the step functions in $\mathfrak{R}(K, V)$ follows by the usual compactness arguments for the interval $[\tau, \mathcal{T}]$ (see, e.g., [Die85, Sect. 7.6] for limits of arbitrary step functions), and equiuniform approximation of continuous functions uses standard arguments well known from Riemann integral. For the strong versions we just apply these arguments to functions of the form Fv in $\mathfrak{R}(K, V)$ and in $\mathcal{C}(K, V)$, respectively. Of course, the statements for K' are just restrictions of the statement for K . ■

We want to define an integral

$$\int_{\tau}^{\mathcal{T}} F_t dI_t G_t := \lim_{P \in \mathcal{P}_K} \sum_{k=1}^N F_{t_{k-1}} dI_{t_k} G_{t_{k-1}}, \quad (5.1)$$

where $F, G \in \mathfrak{R}^s(K, \mathcal{B})$ and I is some function $K \rightarrow \mathcal{B}$ and $dI_{t_k} = I_{t_k} - I_{t_{k-1}}$. Suppose F and G are step functions, i.e., $F = F^{P_F}$ and $G = G^{P_G}$ for suitable $P_F, P_G \in \mathcal{P}_K$. Then

$$\int_{\tau}^{\mathcal{T}} F_t dI_t G_t = \sum_{k=1}^N F_{t_{k-1}} dI_{t_k} G_{t_{k-1}}$$

for all partitions $P \geq \max(P_F, P_G)$. The following proposition is a simple consequence of Proposition 5.2.

5.3. PROPOSITION. *Suppose*

$$(F, G) \mapsto \int_{\tau}^{\mathcal{T}} F_t dI_t G_t$$

is bounded on $\mathfrak{S}(K, \mathcal{B}) \times \mathfrak{S}(K, \mathcal{B})$. Then (5.1) exists

- (1) as an equiuniform limit on $\mathcal{C}(K, \mathcal{B}) \times \mathcal{C}(K, \mathcal{B})$;
- (2) as an equistrong limit on $\mathcal{C}(K, \mathcal{B}) \times \mathcal{C}^s(K, \mathcal{B})$;
- (3) as a uniform limit on $\mathfrak{R}(K, \mathcal{B}) \times \mathfrak{R}(K, \mathcal{B})$;
- (4) as a strong limit on $\mathfrak{R}(K, \mathcal{B}) \times \mathfrak{R}^s(K, \mathcal{B})$.

So far we introduced the spaces \mathcal{C} , \mathcal{C}^s , \mathfrak{R} , and \mathfrak{R}^s which are related to the integrands in our integrals. In Proposition 5.3 we needed boundedness on step function, but we did not specify further properties of the integrator I_t .

Now we introduce the spaces related to I_t . These spaces are \mathfrak{Q}^∞ -spaces and, therefore, more related to measure theoretic methods. For an integral over an interval K it is sufficient that the integrator is bounded only on K . This leads to the well-known notions of spaces of locally bounded functions.

5.4. DEFINITION. Let $K = [\tau, \mathcal{T}] \subset \mathbb{R}$ be a compact interval and let V be a Banach space. A function $F: K \rightarrow V$ (or $F: \mathbb{R} \rightarrow V$) is called *simple*, if it is measurable and finitely valued, i.e., if it is a finite sum of functions of the form $\chi_S v$ for S being a measurable subset of K and $v \in V$. Obviously, a simple function is bounded.

By $\mathfrak{Q}^\infty(K, V)$ we denote the space of *bounded* V -valued functions on K , i.e., the completion of the simple function in the norm $\|\bullet\|^K$. By $\mathfrak{Q}^{\infty, s}(K, \mathcal{B})$ we denote the space of *strongly bounded* \mathcal{B} -valued functions on K , i.e., the completion of the simple function in the *strong topology* which is generated by the family $\|\bullet\|_v^K$ ($v \in V$) of seminorms. By the *principle of uniform boundedness*, $\|\bullet\|^K$ is finite for all $F \in \mathfrak{Q}^{\infty, s}(K, \mathcal{B})$ so that $\mathfrak{Q}^{\infty, s}(K, \mathcal{B})$ is a Banach space, too.

By $\mathfrak{Q}_{\text{loc}}^\infty(V)$ we denote the space of *locally bounded* V -valued functions on \mathbb{R} , i.e., all functions $F: \mathbb{R} \rightarrow V$ such that $F \upharpoonright K \in \mathfrak{Q}^\infty(K, V)$ for K . We endow $\mathfrak{Q}_{\text{loc}}^\infty(V)$ with the corresponding projective limit topology (cf. Definition 6.5). The space $\mathfrak{Q}_{\text{loc}}^{\infty, s}(V)$ of *locally strongly bounded* \mathcal{B} -valued functions on \mathbb{R} is defined in a similar manner.

5.5. Remark. In all our applications functions in \mathfrak{Q}^∞ -spaces give rise to non-zero contributions, only if the semi-norm $\text{ess sup } \|\bullet\|$ is non-zero; cf. Section 8. Therefore, we could also change to the well-known L^∞ -spaces of *essentially bounded* functions, which arise from the corresponding \mathfrak{Q}^∞ -spaces by dividing out the kernel of the semi-norm $\text{ess sup } \|\bullet\|$. However, a strong version of this is delicate due to measurability problems. Therefore, with one exception in Section 8, we will stay with \mathfrak{Q}^∞ -spaces.

Suppose $I': K \rightarrow \mathcal{B}$ is simple where $K = [\tau, \mathcal{T}]$ is a compact interval. Then we may find finitely many measurable sets $S_i \subset K$ and operators $a_i \in \mathcal{B}$ ($i = 1, \dots, n$) such that $I' = \sum_{i=1}^n \chi_{S_i} a_i$. Moreover, we may assume the

S_i to be pairwise disjoint. Obviously, $\|I'\|^K = \max(\|a_i\|)$. Let λ denote the Lebesgue measure on \mathbb{R} . For $\tau \leq \mathcal{T}$ we define

$$\int_{\tau}^{\mathcal{T}} I'_t dt = \sum_{i=1}^n \lambda(S_i \cap [\tau, \mathcal{T}]) a_i.$$

We easily find $\|\int_{\tau}^{\mathcal{T}} I'_t dt\|^K \leq (\mathcal{T} - \tau) \max(\|a_i\|) = (\mathcal{T} - \tau) \|I'\|^K$. Therefore, we may extend the definition of the integral to functions $I' \in \mathfrak{L}^{\infty}(K, \mathcal{B})$ by norm continuity, and, using a similar estimate, to functions $I' \in \mathfrak{L}^{\infty, s}(K, \mathcal{B})$ by strong continuity. The integral extends further to functions $I' \in \mathfrak{L}_{\text{loc}}^{\infty}(\mathcal{B})$ and $I' \in \mathfrak{L}_{\text{loc}}^{\infty, s}(\mathcal{B})$, respectively, by first restricting I' to K .

5.6. DEFINITION. We say an integrator function $I: \mathbb{R} \rightarrow \mathcal{B}$ has a locally bounded and a locally strongly bounded density $I' \in \mathfrak{L}_{\text{loc}}^{\infty}(\mathcal{B})$ and $I' \in \mathfrak{L}_{\text{loc}}^{\infty, s}(\mathcal{B})$, respectively, if

$$I_t - I_0 = \begin{cases} \int_0^t I'_s ds & \text{for } t \geq 0 \\ -\int_t^0 I'_s ds & \text{otherwise.} \end{cases}$$

5.7. PROPOSITION. Suppose I has a locally bounded density I' . Then (5.1) exists

- (1) as an equiuniform limit on $\mathcal{C}(K, \mathcal{B}) \times \mathcal{C}(K, \mathcal{B})$;
- (2) as an equistrong limit on $\mathcal{C}^s(K, \mathcal{B}) \times \mathcal{C}^s(K, \mathcal{B})$;
- (3) as a uniform limit on $\mathfrak{R}(K, \mathcal{B}) \times \mathfrak{R}(K, \mathcal{B})$;
- (4) as a strong limit on $\mathfrak{R}^s(K, \mathcal{B}) \times \mathfrak{R}^s(K, \mathcal{B})$.

In all cases we have

$$\int_{\tau}^{\mathcal{T}} F_t dI_t G_t = \int_{\tau}^{\mathcal{T}} F_t I'_t G_t dt, \quad (5.2)$$

where, obviously, $FI'G \in \mathfrak{L}_{\text{loc}}^{\infty}(\mathcal{B})$ and $FI'G \in \mathfrak{L}_{\text{loc}}^{\infty, s}(\mathcal{B})$, respectively. Formally, we write $dI_t = I'_t dt$.

5.8. Remark. We call $\int_{\tau}^{\mathcal{T}} F_t dI_t G_t$ a generalized time integral. The difference between $\int_{\tau}^{\mathcal{T}} F_t dI_t G_t$ and $\int_{\tau}^{\mathcal{T}} F_t I'_t G_t dt$ lies in the Riemann–Stieltjes sums which are suggested by the respective integral. Proposition 5.7 tells us that $\int_{\tau}^{\mathcal{T}} F_t dI_t G_t$ is a limit of the associated Riemann–Stieltjes sums. For $\int_{\tau}^{\mathcal{T}} F_t I'_t G_t dt$, in general, this is not true, already in the scalar case.

Proof of Proposition 5.7. The proof is based on the rude estimate

$$\left\| \sum_{k=1}^N F_{t_{k-1}} dI_{t_k} G_{t_{k-1}} \right\| \leq (\mathcal{T} - \tau) \|I'\|^K \|F\|^K \|G\|^K \quad (5.3)$$

which holds for arbitrary functions $F, G: \mathbb{R} \rightarrow \mathcal{B}$ and all partitions $P \in \mathcal{P}_K$. By (5.3) we may assume that I' is simple. (Otherwise, replace it by a simple function sufficiently close to I' in $\|\bullet\|^K$.) As simple functions are finite sums over functions of the form $\chi_S a$, we even may assume $I' = \chi_S a$. Thus, we are reduced to the case

$$\sum_{k=1}^N F_{t_{k-1}} dI_{t_k} G_{t_{k-1}} = \sum_{k=1}^N d\tilde{I}_{t_k} F_{t_{k-1}} a G_{t_{k-1}}$$

with $\tilde{I} = \chi_S$. It remains to mention that $FaG \in \mathcal{C}^s(K, \mathcal{B})$, whenever $F, G \in \mathcal{C}^s(K, \mathcal{B})$, and (by similar arguments) that $FaG \in \mathcal{R}^s(K, \mathcal{B})$, whenever $F, G \in \mathcal{R}^s(K, \mathcal{B})$. Then the desired convergences follow by Proposition 5.2 as for the usual Riemann integral. Equation (5.2) follows from the observation that the integral on the right-hand side is approximated by simple functions of the form $F^P I' G^P$ (I' simple) in the respective topologies. ■

6. INTEGRALS

In this section we define the $*$ -algebra \mathfrak{U}_1 of adapted processes and define for them stochastic integrals with respect to creation, annihilation, conservation processes, and the time integral. We use, however, a condensed notation where, formally, only conservation integrals appear, however, where the class of processes which are allowed as integrands is bigger. This condensed notation does not contain more or less information. It allows, however, for more economic proofs. Moreover, a generalization to higher order integrals is possible.

6.1. DEFINITION. Let \mathbf{B} be a unital C^* -algebra and let E be a \mathbf{B} - \mathbf{B} -Hilbert module. By $L^2(\mathbb{R}, E)$ we denote the *exterior tensor product* of E and the \mathbb{C} - \mathbb{C} -Hilbert module $L^2(\mathbb{R})$, i.e., the Hilbert module completion of the algebraic tensor product $E \otimes L^2(\mathbb{R})$ with inner product $\langle x \otimes f, y \otimes g \rangle = \langle x, y \rangle \langle f, g \rangle$. For $-\infty < t \leq \infty$ we denote $E_t = L^2((-\infty, t), E) \subset L^2(\mathbb{R}, E) = E_\infty$.

We work on the full Fock module $\mathcal{F} = \mathcal{F}(E_\infty)$. By $\mathcal{F}^{(n)} = E_\infty^{\odot n}$ ($n \in \mathbb{N}_0$) we denote the n -particle sector. We decompose $\mathcal{B}^a(\mathcal{F})$ into the *homogeneous subspaces* $\mathcal{B}^{(n)} = \{b \in \mathcal{B}^a(\mathcal{F}) : b\mathcal{F}^{(m)} \subset \mathcal{F}^{(m+n)} (m \in \mathbb{Z})\}$ ($n \in \mathbb{Z}$). By

\mathcal{B}_g we denote the algebraic direct sum over all $\mathcal{B}^{(n)}$ and by \mathcal{B}_1 we denote its completion with respect to the ℓ^1 -norm $\|\bullet\|_1$ as defined in Section 3.

6.2. *Remark.* As the step functions $\mathfrak{S}(\mathbb{R})$ are dense in $L^2(\mathbb{R})$, the step functions $\mathfrak{S}(\mathbb{R}, E)$ are dense in $L^2(\mathbb{R}, E)$. As $\chi_S \in L^2(K)$ for any measurable set $S \subset K$, the simple functions $K \rightarrow E$ are contained in $L^2(K, E)$. Observe that for simple functions $x \in L^2(K, E)$

$$\|x\| \leq \sqrt{\mathcal{F} - \tau} \|x\|^K$$

(cf. the discussion before Definition 5.6). Therefore, $\mathfrak{Q}^\infty(K, E) \subset L^2(K, E)$. (More precisely, given a sequence (x_n) of simple functions in $\mathfrak{Q}^\infty(K, E)$ converging to some $x \in \mathfrak{Q}^\infty(K, E)$, we may find a sequence (s_n) of step functions in $\mathfrak{Q}^\infty(K, E)$ fulfilling $\|x_n - s_n\|^K < \frac{1}{n}$ ($n \in \mathbb{N}$). Then the sequence (s_n) is a Cauchy sequence in $L^2(K, E)$ and its limit depends neither on the choice of (x_n) , nor on the choice of (s_n) , but only on the choice of x . Of course, the identification is only almost everywhere. In particular, it is not injective. However, notice that this identification becomes injective immediately, if we pass to $L^\infty(K, E)$ instead of $\mathfrak{Q}^\infty(K, E)$; cf. Remark 5.5.) Moreover, the elements of $\mathfrak{Q}^\infty(K, \mathcal{B}^a(E))$ act as bounded operators on $L^2(K, E)$ and leave invariant $\mathfrak{Q}^\infty(K, E)$.

As explained in the beginning, we express all our integrals in a way that they formally look like conservation integrals. So we use only one integrator function $p \in \mathfrak{Q}_{\text{loc}}^{\infty, s}(\mathcal{B}_1)$ with $p_t = p(\chi_{(-\infty, t]})$. The form of the Riemann–Stieltjes sums as in (5.1) with two processes as integrands reminds us of an inner product with values in $\mathcal{B}^a(\mathcal{F})$. We make this explicit.

Let $K = [\tau, \mathcal{T}]$ be compact interval and let $P \in \mathcal{P}_K$. By setting

$$(F, G)_P = \sum_{k=1}^N F_{t_{k-1}} dp_{t_k} G_{t_{k-1}},$$

we define a $\mathcal{B}^a(\mathcal{F})$ -valued, $\mathcal{B}^a(\mathcal{F})$ – $\mathcal{B}^a(\mathcal{F})$ -linear (i.e., $(aF, Ga')_P = a(F, G)_P a'$ for all $a, a' \in \mathcal{B}^a(\mathcal{F})$) mapping on the $\mathcal{B}^a(\mathcal{F})$ – $\mathcal{B}^a(\mathcal{F})$ -module of all mappings $\mathbb{R} \rightarrow \mathcal{B}^a(\mathcal{F})$. By the following lemma this mapping is *positive* (i.e., $(F^*, F)_P \geq 0$) so that we may speak of a bilinear (not a sesquilinear) inner product. Of course, $(F, G)_P^* = (G^*, F^*)_P$.

6.3. *LEMMA.* For all functions $F, G: \mathbb{R} \rightarrow \mathcal{B}^a(\mathcal{F})$ we have

$$(F, G)_P = (F, \mathbf{1})_P (\mathbf{1}, G)_P. \quad (6.1)$$

Proof. This follows immediately from $dp_{t_k} dp_{t_\ell} = dp_{t_k} \delta_{k\ell}$. ▀

6.4. COROLLARY. *We have the Cauchy–Schwarz inequality*

$$|(F, G)_P|^2 = (\mathbf{1}, G)_P^* (F, \mathbf{1})_P^* (F, \mathbf{1})_P (\mathbf{1}, G)_P \\ \leq \|(F, \mathbf{1})_P^* (F, \mathbf{1})_P\| (\mathbf{1}, G)_P^* (\mathbf{1}, G)_P = \|(F, F^*)_P\| (G^*, G)_P.$$

Lemma 6.3 may be considered as a particularly simple example for an *Ito formula*. We see that in order to analyze under which circumstances the two-sided integral $\int F_t dp_t G_t$ exists, it is sufficient to understand under which circumstances the one-sided integrals $\int dp_t G_t$ and $\int F_t dp_t$ exist. Of course, the two types are adjoints of each other (put $F = G^*$). Therefore, if we show existence of both one-sided integrals as a strong limit, actually, we show that both exist as $*$ -strong limits. If, additionally, the nets $(F, \mathbf{1})_P$ and $(\mathbf{1}, G)_P$ are bounded, then also the net $(F, G)_P$ converges $*$ -strongly.

Lemma 6.3 holds for arbitrary processes F, G . In order to show convergence of the inner product $(F, G)_P$, we have to restrict our processes to smaller classes.

6.5. DEFINITION. The $*$ -algebra of processes \mathfrak{P} consists of all families $F = (F_t)_{t \in \mathbb{R}}$ of elements $F_t \in \mathcal{B}^a(\mathcal{F})$ which are $*$ -strongly continuous as mappings $t \mapsto F_t$.

Let K be a compact interval. The set $\mathfrak{P}^K = \{F \upharpoonright K : F \in \mathfrak{P}\}$ is nothing but the C^* -algebra $\mathcal{C}^s(K, \mathcal{B}^a(\mathcal{F})) \cap \mathcal{C}^s(K, \mathcal{B}^a(\mathcal{F}))^*$ and \mathfrak{P} is the *projective limit* of all \mathfrak{P}^K . We equip \mathfrak{P} with the *projective limit topology* arising from this projective limit. In other words, a net of elements F in \mathfrak{P} converges, if and only if the net of restrictions $F \upharpoonright K$ converges in \mathfrak{P}^K for each compact interval K . We also say \mathfrak{P} consists of all *locally bounded $*$ -strongly continuous functions* $F: \mathbb{R} \rightarrow \mathcal{B}^a(\mathcal{F})$.

We decompose \mathfrak{P} into the *homogeneous subspaces* $\mathfrak{P}^{(n)} = \{F \in \mathfrak{P} : F_t \in \mathcal{B}^{(n)}(t \in \mathbb{R})\}$ ($n \in \mathbb{Z}$). By \mathfrak{P}_g we denote the algebraic direct sum over all $\mathfrak{P}^{(n)}$ and by \mathfrak{P}_1 we denote its completion with respect to the ℓ -norm $\|\bullet\|_1$ as defined in Section 3. We use similar notations for \mathfrak{P}^K .

The $*$ -algebra of adapted processes \mathfrak{U} consists of all $F \in \mathfrak{P}$ such that F_t is adapted to E_t . By $\mathfrak{U}^\tau \subset \mathfrak{P}$ we denote the $*$ -algebra of those processes where F_t is adapted to E_t at least for $t \geq \tau$. We set $\mathfrak{U}^{(n)} = \mathfrak{U} \cap \mathfrak{P}^{(n)}$, $\mathfrak{U}_g = \mathfrak{U} \cap \mathfrak{P}_g$, and $\mathfrak{U}_1 = \mathfrak{U} \cap \mathfrak{P}_1$. We use similar notations for \mathfrak{U}^K and \mathfrak{U}^τ .

We are interested in showing existence of the following four limits over \mathcal{P}_K . First, $(F, G)_P$ where F, G are adapted. This corresponds to the usual conservation integral. In order to include also an argument $T \in \mathfrak{Q}_{\text{loc}}^\infty(\mathcal{B}^a(E))$ for the integrator, we consider the slightly more general $(F, p(T) G)_P = (Fp(T), G)_P$. Second, $(F, \ell^*(\chi_K x) G)_P$ where $x \in \mathfrak{Q}_{\text{loc}}^\infty(E)$ (so that $\chi_K x \in E_\infty$ by Remark 6.2), and, third, its adjoint. These correspond to the usual

creation integral and annihilation integral, respectively. Fourthly, $(F\ell(\chi_K x), \ell^*(\chi_K y) G)_P$ where $x, y \in \mathfrak{Q}_{\text{loc}}^\infty(E)$. This corresponds to the integral with respect to the operator valued measure $\mu^{x,y}([s, t]) = \langle x, \chi_{[s, t]} y \rangle$. One easily checks that

$$\ell(\chi_K x) p_t \ell^*(\chi_K y) = \int_\tau^t \langle x(s), y(s) \rangle ds = \langle x, \chi_{[\tau, t]} y \rangle$$

($t \in K$) where by Cauchy–Schwarz inequality $\langle x(t), y(t) \rangle \chi_K(t)$ is a bounded function. In other words, the assumptions of Proposition 5.7 are fulfilled and we are concerned with a generalized time integral.

6.6. COROLLARY. *The integral*

$$\int_\tau^\mathcal{T} F_t d\mu_t^{x,y} G_t := \lim_{P \in \mathfrak{P}_K} (F\ell(\chi_K x), \ell^*(\chi_K y) G)_P$$

exists *-equistrongly for all $F, G \in \mathfrak{B}$, and it coincides with $\int_\tau^\mathcal{T} F_t \langle x(t), y(t) \rangle G_t dt$. Moreover,

$$\|(F\ell(\chi_K x), \ell^*(\chi_K y) G)_P\| \leq (\mathcal{T} - \tau) \|x\|^K \|y\|^K \|F\|^K \|G\|^K$$

and, in particular,

$$\|(\mathbf{1}, \ell^*(\chi_K x) F)_P\| = \|(F^* \ell(\chi_K x), \mathbf{1})_P\| \leq \sqrt{\mathcal{T} - \tau} \|x\|^K \|F\|^K \quad (6.2)$$

for all $F, G: \mathbb{R} \rightarrow \mathcal{B}^a(\mathcal{F})$.

6.7. LEMMA. *Let $F: \mathbb{R} \rightarrow \mathcal{B}^{(n)}$ ($n \in \mathbb{Z}$) be an adapted function. Then*

$$\|(F, \mathbf{1})_P\| = \|(\mathbf{1}, F^*)_P\| \leq \|F\|^K.$$

Proof. Observe that $F_t^* F_t \in \mathcal{B}^{(0)}$. Therefore, by Corollary 4.7 and the relations in Proposition 3.4 we have

$$\begin{aligned} \left| \sum_{k=1}^N F_{t_{k-1}} dp_{t_k} \right|^2 &= \sum_{k, \ell=1}^N dp_{t_k} F_{t_{k-1}}^* F_{t_{\ell-1}} dp_{t_\ell} = \sum_{k, \ell=1}^N dp_{t_k} \mathbb{E}_0(F_{t_{k-1}}^* F_{t_{\ell-1}}) dp_{t_\ell} \\ &= \sum_{k=1}^N dp_{t_k} \mathbb{E}_0(F_{t_{k-1}}^* F_{t_{k-1}}) dp_{t_k} \leq (\|F\|^K)^2 \sum_{k=1}^N dp_{t_k} \leq (\|F\|^K)^2. \quad \blacksquare \end{aligned}$$

6.8. PROPOSITION. *All Riemann–Stieltjes sums $(\bullet, \bullet)_P$ are bounded in ℓ^1 -norm on the considered class of processes. More precisely, let $F, G \in \mathfrak{U}_1^\tau$, $x, y \in \mathfrak{Q}_{\text{loc}}^\infty(E)$, and $T \in \mathfrak{Q}_{\text{loc}}^\infty(\mathcal{B}^a(E))$. Then*

$$\|(F, p(T) G)_P\|_1 = \|(Fp(T), G)_P\|_1 \leq \|T\|^K \|F\|_1^K \|G\|_1^K \quad (6.3a)$$

$$\begin{aligned} \|(F, \ell^*(\chi_K x) G)_P\|_1 &= \|(G^* \ell(\chi_K x), F^*)_P\|_1 \\ &\leq \sqrt{\mathcal{T} - \tau} \|x\|^K \|F\|_1^K \|G\|_1^K \end{aligned} \quad (6.3b)$$

$$\|(F\ell(\chi_K x), \ell^*(\chi_K y), G)_P\|_1 \leq (\mathcal{T} - \tau) \|x\|^K \|y\|^K \|F\|_1^K \|G\|_1^K. \quad (6.3c)$$

Proof. By Lemma 2.1 it is sufficient to show the estimates for homogeneous processes F, G . By Corollary 6.4 and Lemma 6.7 we find

$$\begin{aligned} \|(F, p(T) G)_P\| &\leq \|F\|^K \|(\mathbf{1}, p(T) G)_P\| = \|F\|^K \|(p(T), G)_P\| \\ &= \|F\|^K \|p(T)(\mathbf{1}, G)_P\| \leq \|T\|^K \|F\|^K \|G\|^K. \end{aligned}$$

This shows (6.3a). Equations (6.3b) and (6.3c) follows in a similar manner from Corollary 6.6. ■

6.9. THEOREM. *Let $F, G \in \mathfrak{U}_1^\tau$, $x, y \in \mathfrak{L}_{\text{loc}}^\infty(E)$, and $T \in \mathfrak{L}_{\text{loc}}^\infty(\mathcal{B}^a(E))$. Then the conservation integral*

$$\int_\tau^\mathcal{T} F_t dp_t(T) G_t := \lim_{P \in \mathfrak{P}_K} (F, p(T) G)_P, \quad (6.4a)$$

the creation integral

$$\int_\tau^\mathcal{T} F_t d\ell_t^*(x) G_t := \lim_{P \in \mathfrak{P}_K} (F, \ell^*(\chi_K x) G)_P, \quad (6.4b)$$

and the annihilation integral

$$\int_\tau^\mathcal{T} F_t d\ell_t(x) G_t := \lim_{P \in \mathfrak{P}_K} (F\ell(\chi_K x), G)_P \quad (6.4c)$$

*exist, like the (generalized) time integral $\int_\tau^\mathcal{T} F_t d\mu_t^{x,y} G_t = \int_\tau^\mathcal{T} F_t \langle x(t), y(t) \rangle G_t dt$, as *-equistrong limits in the *-strong topology of \mathcal{B}_1 .*

Moreover, for all four integrators the process M defined by setting

$$M_t = \begin{cases} \int_\tau^t F_s dI_s G_s & \text{for } t \geq \tau \\ 0 & \text{otherwise} \end{cases} \quad (6.5)$$

is an element of \mathfrak{U}_1^τ .

Proof. By Proposition 6.8 the assumptions of Lemma 2.2 are fulfilled so that we may reduce to homogeneous elements. Moreover, all nets are bounded. Therefore, as explained after Corollary 6.4, it is sufficient to show strong convergence in each of the cases $(\mathbf{1}, G)_P$ and $(\mathbf{1}, \ell^*(\chi_K x) G)_P$ and the respective adjoints. (Of course, the case $(\mathbf{1}, p(T) G)_P = p(T)(\mathbf{1}, G)_P$ is included in the case $(\mathbf{1}, G)_P$.)

By Lemma 4.5, in $(\mathbf{1}, G)_P$ we may replace G by the process $\hat{\ell}(G^* \omega)$ which is $\|\bullet\|_1$ -continuous by Corollary 3.9. Therefore, by Proposition 6.8 we are in the first case of Proposition 5.3. This even settles norm convergence of both $(\mathbf{1}, G)_P$ and $(G^*, \mathbf{1})_P$.

Strong convergence of $(\mathbf{1}, \ell^*(\chi_K x) G)_P$ is settled by the strong analogue

$$\|(\mathbf{1}, \ell^*(\chi_K x) G)_P Z\| \leq \sqrt{\mathcal{T} - \tau} \|x\|^K \|G\|_Z^K$$

of (6.2) for all Z in the whole domain \mathcal{F} .

For the case $(G^* \ell(\chi_K x), \mathbf{1})_P$ we choose $Z = z \odot Z'$ where $z \in \mathfrak{L}_{\text{loc}}^\infty(E) \cap E_\infty$, and $Z' \in \mathcal{F}$. We find

$$\ell(\chi_K x) dp_{t_k} Z = \langle \chi_K x, d\chi_{t_k} z \rangle Z'.$$

Therefore, by Corollary 6.6

$$(G^* \ell(\chi_K x), \mathbf{1})_P Z = (G^* \ell(x), \ell^*(z) \mathbf{1})_P Z \rightarrow \int_\tau^{\mathcal{T}} G_t^* d\mu_t^{x, z} \mathbf{1} Z' \quad (6.6)$$

equiuniformly. Since the net $(G^* \ell(x), \mathbf{1})_P$ is bounded, and since the $z \odot Z'$ form a total subset of \mathcal{F} , we obtain equistrong convergence on \mathcal{F} .

Clearly, M_t is adapted to E_t . And, clearly, by Proposition 6.8 the time, creation, and annihilation integrals depend even continuously on their upper bound. To see strong continuity (once again, this is sufficient by symmetry under adjoint) of the conservation integral, we also may consider $\int_\tau^{\mathcal{T}} G_t^* dp_t$ and $\int_\tau^{\mathcal{T}} dp_t G_t$ separately. The former case is clear by (6.6). For the latter the idea is the same, but we need a more refined argument. We choose $G \in \mathfrak{U}^{(-n)}$ where $n \geq 0$ (otherwise $\hat{\ell}(G^* \omega) = 0$), and $Z = Z^{(n)} \odot z \odot Z'$ where $Z^{(n)} \in \mathcal{F}^{(n)}$ and z, Z' as before. We find

$$\begin{aligned} \|(\mathbf{1}, G)_P Z\|^2 &= \|\langle Z, (G^*, G)_P Z \rangle\| \\ &= \|\langle Z', (\zeta^* \ell(\chi_K z), \ell^*(\chi_K z) \zeta)_P Z' \rangle\| \leq (\mathcal{T} - \tau) (\|\zeta\|^K)^2 \|Z'\|^2 \end{aligned}$$

for all $P \in \mathcal{P}_K$, where ζ is the adapted process $t \mapsto \langle \omega, G_t Z^{(n)} \rangle \in \mathbf{B} \subset \mathcal{B}_1$. For P sufficiently fine, $(\mathbf{1}, G)_P Z$ is close to $\int_\tau^{\mathcal{T}} dp_t G_t Z$. This implies strong continuity on a total subset, hence, everywhere. ■

6.10. *Remark.* In the sequel, we will use shorthand notations like

$$\int_{\tau}^{\mathcal{T}} F dI G = \int_{\tau}^{\mathcal{T}} F_t dI_t G_t, \quad \int_{\tau}^t F dI G = M_t, \quad \text{and} \quad \int F dI G = M,$$

if no confusion can arise. But keep in mind that $M_t = 0$ for $t \leq \tau$.

6.11. *Remark.* As the proof shows, many statements in Theorem 6.9 can be specified further. Additionally, weakening the convergence to *-strong convergence, all integrals exist also if the processes are only *-strong càdlàg functions.

Conversely, if we restrict to continuous integrands, then also the creation, annihilation, and time integral converge in norm. Therefore, if we omit the conservation integral (which is essentially non-continuous; see Lemma 8.4), then we may restrict as in [KS92] to a theory of continuous processes where everything converges in norm.

We also mention that for most statements it is not necessary to factorize according to Lemma 6.3. We emphasize, however, that convergence of the annihilation integral becomes much more complicated without this factorization.

6.12. *Remark.* In [Lie98] Liebscher considers a generalization of the usual conservation integral in the calculus on the symmetric Fock space by Hudson and Parthasarathy [HP84, Par92]. In this generalization the conservation integral is explained not only for time functions T , but for all operators $T \in \mathcal{B}^a(E_{\infty})$. Unlike the usual behaviour in symmetric calculus, the integrators do no longer commute with the processes. Consequently, in [Lie98] there are two types of conservation integrals, one with the process on the right of the integrator, and one with the process on the left. One of the two possibilities is so complicated that its existence is guaranteed (explicitly) only for simple integrands.

A literal translation into our free setup encourages to consider limits of $(p(T), F)_P$ and of $(F, p(T))_P$. However, by the particularly simple rules in Proposition 3.4 we find

$$(p(T), F)_P = p(T)(\mathbf{1}, F)_P \quad \text{and} \quad (F, p(T))_P = (F, \mathbf{1})_P p(T).$$

Convergence of these expressions becomes a trivial application of Theorem 6.9. We ask two questions. First, could it be possible to translate this back into the symmetric framework? Second, is it possible to treat limits of expressions with two integrands like $(Fp(T), G)_P$ and $(F, p(T)G)_P$? (Of course, we still have $(F, p(T)G)_P = (F, \mathbf{1})_P (\mathbf{1}, p(T)G)_P$. However, as $p(T)$ does no longer commute with dp_t , we cannot treat $(\mathbf{1}, p(T)G)_P$ as before.) Presently, we do not know the answers.

7. DIFFERENTIAL EQUATIONS

In this section we show that a quite general class of quantum stochastic differential equations has unique solutions. A typical differential equation has the form

$$dW = W dM, \quad M_\tau = w, \quad (7.1)$$

where $dM = F^0 dI G^0 + F^+ d\ell^*(x) G^+ + F^- d\ell(y) G^- + F^1 dp(T) G^1$ (as in Theorem 9.4 below) and w is an operator on \mathcal{F} adapted to E_τ . (Of course, also the adjoint equation is considered.) A solution of such a differential equation is a process $W \in \mathfrak{U}_1^\tau$ fulfilling

$$W_t = w + \int_\tau^t W dM.$$

The standard procedure already used in the calculus on the symmetric Fock space [HP84] is *successive approximation*. We also follow this approach. However, thanks to the fact that we are dealing with bounded operators, we are able as in [KS92, Spe91] to show convergence by an application of *Banach's fix point theorem*. As in [KS92] for a calculus without conservation integrals we may apply the fix point theorem directly. If conservation integrals are involved, we need a triple iteration (cf. [Spe91]). In both cases we will meet more general types of differential equations, when we consider unitarity conditions. Therefore, we decided to keep the description from the beginning as general as possible.

7.1. DEFINITION. A *general integral* is a linear mapping $\mathcal{J}: \mathfrak{U}_1^\tau \rightarrow \mathcal{B}^a(\mathcal{F})$ which is contained in the linear span of mappings of one of the forms

$$W \mapsto \int_\tau^\mathcal{T} WF dI G, \quad W \mapsto \int_\tau^\mathcal{T} \mathbb{E}_0(W) F dI G$$

$$W \mapsto \int_\tau^\mathcal{T} FW dI G, \quad W \mapsto \int_\tau^\mathcal{T} F \mathbb{E}_0(W) dI G$$

$$W \mapsto \int_\tau^\mathcal{T} F dI WG, \quad W \mapsto \int_\tau^\mathcal{T} F dI \mathbb{E}_0(W) G$$

$$W \mapsto \int_\tau^\mathcal{T} F dI GW, \quad W \mapsto \int_\tau^\mathcal{T} F dI G \mathbb{E}_0(W),$$

where dI is one of the integrators dl , $d\ell^*(x)$, $d\ell(x)$, or $dp(T)$ ($l' \in \mathfrak{L}_{\text{loc}}^\infty(\mathbf{B})$, $x \in \mathfrak{L}_{\text{loc}}^\infty(E)$, $T \in \mathfrak{L}_{\text{loc}}^\infty(\mathcal{B}^a(E))$ and $F, G \in \mathfrak{U}_1$, or

$$W \mapsto \int_{\tau}^{\mathcal{T}} F dI^{\mathbb{E}_0(W)} G,$$

where the argument of the integrator $dI^{\mathbb{E}_0(W)}$ depends linearly (or anti-linearly for the annihilator) and continuously (in the respective norms) on $\mathbb{E}_0(W)$. We write $\mathcal{J}_{\tau}^{\mathcal{T}}$, if we want to indicate the end points of the involved time interval. By $\mathcal{J}_{\tau}(W)$ we denote the process $t \mapsto \mathcal{J}_{\tau}^t(W)$

A *special general integral* is a general integral where the appearing conservation integrals are subject to the restriction that the parameters F, G in each conservation integral take values only in $\mathbf{B} \subset \mathcal{B}^a(\mathcal{F})$.

The definition of a general integral is motivated by the way processes enter the Ito formula (cf. (9.1) below). Whereas the restriction for the special general integral is necessary, if we want to apply the following refined version of *Banach's fix point theorem*. Already in the calculus on the full Fock space Speicher [Spe91] has shown that there exist differential equations with general conservation integrals which do not have a solution even, if we allow for unbounded operators.

7.2. PROPOSITION. *Let \mathcal{J} be a general integral. Assume that for each compact interval K there exist constants $0 < C < 1$ and $d > 0$ such that*

$$\|\mathcal{J}_t^{t+\delta} \circ \mathcal{J}_t \circ \mathcal{J}_t(W)\|_1 \leq C \|W\|_1$$

for all $t \in K$ and $0 \leq \delta \leq d$. Then for all $\tau \in \mathbb{R}$ and $w \in \mathcal{B}^a(\mathcal{F})$ adapted to E_{τ} the differential equation

$$W_t = w + \mathcal{J}_{\tau}^t(W) \tag{7.2}$$

has a unique solution in \mathfrak{U}_1^{τ} .

Proof. For $s \in [t, t+d]$ and W_t adapted to E_t we find a solution by successive approximation, i.e., we set $W_s^0 = W_t$ and $W_s^{n+1} = W_t + \mathcal{J}_t^s(W^n)$ for $n \geq 1$. Then as in the proof of Banach's fix point theorem the W_s^n form a Cauchy sequence in ℓ^1 -norm whose limit is the unique solution (7.2) on $[t, t+d]$. By splitting a compact interval into finitely many intersecting intervals of length d , we construct a unique solution on each compact interval K . In this way, we obtain for each $t \in \mathbb{R}$ a solution on $[\tau, t]$. By uniqueness the solution restricted to a subinterval $[\tau, s]$ must coincide with the solution constructed on this subinterval so that we obtain a unique solution on $[\tau, \infty)$. Finally, we extend this solution by the constant w to times smaller than τ and obtain a solution on \mathbb{R} which is by construction in \mathfrak{U}_1^{τ} . ■

7.3. THEOREM. *Let \mathcal{J} be a special general integral. Then the differential Eq. (7.2) with $w \in \mathcal{B}^a(\mathcal{F})$ adapted to E_τ has a unique solution in \mathfrak{U}_1^τ .*

Proof. We show that the assumptions of Proposition 7.2 are fulfilled. By Lemma 2.1 it is enough to understand this for each of the (finitely many) homogeneous parts of the operator $\mathcal{J}_t^{t+\delta} \circ \mathcal{J}_t \circ \mathcal{J}_t: \mathfrak{U}_1^\tau \rightarrow \mathcal{B}_1$ and for homogeneous W . In the iterated integral $\mathcal{J}_t^{t+\delta} \circ \mathcal{J}_t \circ \mathcal{J}_t(W)$ we have two types of summands. Either at least one time, creation, or annihilation integral is involved. Then existence of suitable constants C, d follows from (6.3b), (6.3c). Or we have an iterated conservation integral. In this case, we conclude from the fact that dp commutes with all functions taking values in \mathbf{B} and from $dp_{t_k} dp_{t_\ell} = 0$ for $k \neq \ell$ that the triple conservation integral is 0. ■

8. SOME 0-CRITERIA

In this section we prepare for Theorem 9.4 which asserts in how far the coefficients in a stochastic differential equation are unique. The main result is Lemma 8.4 which tells us that conservation integrals are essentially strongly continuous. This allows us to separate them from the other types of integrals in Theorem 6.9 (which are continuous by Proposition 6.8) by looking at their continuity properties.

All results in this section, besides Proposition 8.5, may be considered as consequences of Lemma 6.3 which by computations as in the the proof of Lemma 6.7 give rise to a particularly simple case of an Ito formula for homogeneous integrands in one-sided conservation integrals. For a full proof of Theorem 9.4 we need the full Ito formula for creation and annihilation integrals. Therefore, it is postponed to the end of the following section.

Recall that

$$\operatorname{ess\,sup}_{t \in K} \|f(t)\| = \inf \{ C > 0 \mid \lambda(\{t \in K : \|f(t)\| > C\}) = 0 \}.$$

Obviously, there exists a $t_0 \in K$ such that $\operatorname{ess\,sup}_{t \in [t_0, t_0 + \delta] \cap K} \|f(t)\| = \operatorname{ess\,sup}_{t \in K} \|f(t)\|$ for all $\delta > 0$. (Otherwise, we could cover K with finitely many open intervals on which the $\operatorname{ess\,sup}$ of f is strictly less than its $\operatorname{ess\,sup}$ on K .) Suppose f^s is a simple function with $\|f - f^s\|^K < \varepsilon$. Then there is a measurable non-null-set $S_\delta \subset [t_0, t_0 + \delta] \cap K$ such that $f_{S_\delta} = f^s \upharpoonright S_\delta$ is constant and $\|f_{S_\delta}\| > \operatorname{ess\,sup}_{t \in K} \|f(t)\| - \varepsilon$. (This follows, because $\operatorname{ess\,sup}$ is a seminorm and the $\operatorname{ess\,sup}$ of a simple function $f = \sum_i \chi_{S_i} f_i$ (S_i pairwise disjoint) is the maximum over all $\|f_i\|$ for which S_i is not a null-set.)

We use the same notations, when K is an arbitrary interval. Of course, existence of t_0 depends on compactness of K .

8.1. COROLLARY. *Let $F \in \mathfrak{L}_{\text{loc}}^\infty(\mathcal{B}^a(E))$. Then the norm $(\leq \infty)$ of F considered as an operator on E_∞ is precisely $\text{ess sup}_{t \in \mathbb{R}} \|F_t\|$.*

8.2. COROLLARY. *$L^\infty(K, \mathcal{B}^a(E))$ is a C^* -subalgebra of $\mathcal{B}^a(L^2(K, E))$ (cf. Remark 5.5).*

Proof. We only have to show completeness. So let (f_n) be a sequence in $\mathfrak{L}^\infty(K, \mathcal{B}^a(E))$ such that the images of f_n in $L^\infty(K, \mathcal{B}^a(E))$ form a Cauchy sequence. By passing to a subsequence, we may assume that $\text{ess sup}_{t \in K} \|f_n(t) - f_m(t)\| \leq \frac{1}{k}$ for all $n, m > k$. Then there exist null-sets $S_{k,n,m}$ such that $\|(f_n - f_m) \chi_{K \setminus S_{k,n,m}}\|^K < \frac{1}{k}$. Hence, $S = \bigcup_{k,n,m} S_{k,n,m}$ is a null-set such that $f_n \chi_{K \setminus S}$ is a Cauchy sequence in $\mathfrak{L}^\infty(K, \mathcal{B}^a(E))$. Clearly, the image in $L^\infty(K, \mathcal{B}^a(E))$ of the limit of this sequence is the limit of the images of f_n in $L^\infty(K, \mathcal{B}^a(E))$, i.e., $L^\infty(K, \mathcal{B}^a(E))$ is complete. ■

We prepare for the main result by providing two criteria which allow us to check whether elements in a C^* -algebra are 0.

8.3. PROPOSITION. *Let a, a', b be elements in a C^* -algebra.*

(1) *$ab = 0$, if and only if $abb^* = 0$.*

(2) *Let $a \geq 0$, and $a' \geq 0$. Then $(a + a')b = 0$, if and only if $ab = 0$ and $a'b = 0$.*

Proof. Of course, $ab = 0 \Rightarrow abb^* = 0$. Conversely, $abb^* = 0 \Rightarrow abb^*a^* = 0 \Rightarrow ab = 0$. This is (1).

Of course, $ab = 0$ and $a'b = 0$ implies $(a + a')b = 0$. So let $a, a' \geq 0$. Then $b^*(a + a')b \geq b^*ab \geq 0$ so that $b^*(a + a')b = 0$ implies $b^*ab = 0 \Rightarrow \sqrt{a}b = 0 \Rightarrow ab = 0$, and, similarly, for a' . This is (2). ■

8.4. LEMMA. *Let $F, G \in \mathfrak{U}_1^\tau$ and $T \in \mathfrak{L}_{\text{loc}}^\infty(\mathcal{B}^a(E))$. Then for the process M defined by setting $M_t = \int_\tau^t F dp(T) G$ the following conditions are equivalent.*

(1) *$M = 0$.*

(2) *M is continuous.*

(3) *$\text{ess sup}_{t \in [\tau, \infty)} \|\mathbb{E}_0(F_t^* F_t) T_t \mathbb{E}_0(G_t G_t^*)\| = 0$.*

Proof. (1) \Rightarrow (2). This is obvious.

(2) \Rightarrow (3). We conclude indirectly. So let us assume that there is a compact interval $K = [\tau, \mathcal{T}]$ ($\mathcal{T} \geq \tau$) such that $C = \text{ess sup}_{t \in K} \|\mathbb{E}_0(F_t^* F_t) T_t \mathbb{E}_0(G_t G_t^*)\| > 0$. By the discussion preceding Corollary 8.1 we may choose $t_0 \in [\tau, \mathcal{T})$ such that

$$\text{ess sup}_{t \in [t_0, t_0 + \delta] \cap K} \|\mathbb{E}_0(F_t^* F_t) T_t \mathbb{E}_0(G_t G_t^*)\| = C \quad (8.1)$$

for all $\delta > 0$. Of course, this implies $\|F\|^{[t_0, t_0+\delta] \cap K} > 0$, $\|G\|^{[t_0, t_0+\delta] \cap K} > 0$, and

$$\operatorname{ess\,sup}_{t \in [t_0, t_0+\delta] \cap K} \|T_t\| \geq \frac{C}{(\|F\|^{[t_0, t_0+\delta] \cap K} \|G\|^{[t_0, t_0+\delta] \cap K})^2} > 0.$$

Necessarily, we have $\|\mathbb{E}_0(F_{t_0}^* F_{t_0})\| > 0$ and $\|\mathbb{E}_0(G_{t_0} G_{t_0}^*)\| > 0$. Otherwise, by continuity of $\mathbb{E}_0(F_t^* F_t)$ and $\mathbb{E}_0(G_t G_t^*)$, we obtain a contradiction to (8.1).

If we choose δ sufficiently small, then the following assertions become true. (For simplicity, we assume $t_0 + \delta \in K$.) First, $\int_{t_0}^{t_0+\delta} dM$ is close to $G_{t_0} p(\chi_{[t_0, t_0+\delta]} T) F_{t_0}$, because the norm of the partition $(t_0, t_0 + \delta)$ is δ , therefore, small. Consequently,

$$p(\chi_{[t_0, t_0+\delta]}) F_{t_0}^* \left(\int_{t_0}^{t_0+\delta} dM \right) G_{t_0}^* p(\chi_{[t_0, t_0+\delta]})$$

is close to $\mathbb{E}_0(F_{t_0}^* F_{t_0}) p(\chi_{[t_0, t_0+\delta]} T) \mathbb{E}_0(G_{t_0} G_{t_0}^*)$.

Second, $\mathbb{E}_0(F_{t_0}^* F_{t_0}) T_t \mathbb{E}_0(G_{t_0} G_{t_0}^*)$ is close to $\mathbb{E}_0(F_t^* F_t) T_t \mathbb{E}_0(G_t G_t^*)$ for all $t \in [t_0, t_0 + \delta]$, because $\mathbb{E}_0(F_t^* F_t)$ and $\mathbb{E}_0(G_t G_t^*)$ are continuous. Therefore, by Corollary 8.1 and Proposition 3.4

$$\begin{aligned} & \|\mathbb{E}_0(F_{t_0}^* F_{t_0}) p(\chi_{[t_0, t_0+\delta]} T) \mathbb{E}_0(G_{t_0} G_{t_0}^*)\| \\ &= \|p(\mathbb{E}_0(F_{t_0}^* F_{t_0})(\chi_{[t_0, t_0+\delta]} T) \mathbb{E}_0(G_{t_0} G_{t_0}^*))\| \\ &= \|\mathbb{E}_0(F_{t_0}^* F_{t_0})(\chi_{[t_0, t_0+\delta]} T) \mathbb{E}_0(G_{t_0} G_{t_0}^*)\| \\ &= \operatorname{ess\,sup}_{t \in [t_0, t_0+\delta]} \|\mathbb{E}_0(F_t^* F_t) T_t \mathbb{E}_0(G_t G_t^*)\| \end{aligned}$$

is close to C . As C does not depend on the choice of δ , $\|M_{t_0+\delta} - M_{t_0}\|$ is bounded below by a non-zero positive number. Therefore, M is not continuous at t_0 .

(3) \Rightarrow (1). Again, we conclude indirectly. So let us assume that $M_t \neq 0$ for some $t > \tau$. We may write $F = \sum_{n \in \mathbb{N}_0} F^{(n)}$ and $G = \sum_{n \in \mathbb{N}_0} G^{(-n)}$. (The components with $n < 0$ do not contribute.)

Observe that $\mathbb{E}_0(F^* F) = \sum_{n \in \mathbb{N}_0} \mathbb{E}_0(F^{(n)*} F^{(n)})$ and, similarly, for $\mathbb{E}_0(G G^*)$. Therefore, by Corollary 8.2 and by Part (2) of Proposition 8.3 it is sufficient to show that the element $\mathbb{E}_0(F^{(n)*} F^{(n)}) T \mathbb{E}_0(G^{(-m)} G^{(-m)*}) \neq 0$ in the C^* -algebra $L^\infty([\tau, t], \mathcal{B}^a(E))$ for some $n, m \in \mathbb{N}_0$.

As $M_t \neq 0$, there exist n and m such that

$$\int_\tau^t F^{(n)} dp(T) G^{(-m)} = \left(\int_\tau^t F^{(n)} dp \right) p(T) \left(\int_\tau^t dp G^{(-m)} \right) \neq 0. \quad (8.2)$$

By Part (1) of Proposition 8.3 we have $\int_{\tau}^t dP F^{(n)*} \int_{\tau}^t F^{(n)} dp(T) G^{(-m)} \times \int_{\tau}^t G^{(-m)*} dp \neq 0$. By computations similar to the proof of Lemma 6.7 we find

$$\begin{aligned}
 & \int_{\tau}^t dp F^{(n)*} \int_{\tau}^t F^{(n)} dp(T) G^{(-m)} \int_{\tau}^t G^{(-m)*} dp \\
 &= \lim_{P \in \mathcal{P}_{[\tau, t]}} (\mathbf{1}, F^{(n)*})_P (F^{(n)}, \mathbf{1})_P p(T) (\mathbf{1}, G^{(-m)})_P (G^{(-m)*}, \mathbf{1})_P \\
 &= \int_{\tau}^t \mathbb{E}_0(F^{(n)*} F^{(n)}) dp(T) \mathbb{E}_0(G^{(-m)} G^{(-m)*}) \\
 &= \int_{\tau}^t dp(\mathbb{E}_0(F^{(n)*} F^{(n)}) T \mathbb{E}_0(G^{(-m)} G^{(-m)*})) \\
 &= p(\chi_{[\tau, t]} \mathbb{E}_0(F^{(n)*} F^{(n)}) T \mathbb{E}_0(G^{(-m)} G^{(-m)*})) \neq 0. \tag{8.3}
 \end{aligned}$$

Equality of the last integral and the integral before follows, because it is true for step functions, and because both $\mathbb{E}_0(F^{(n)*} F^{(n)})$ and $\mathbb{E}_0(G^{(-m)} G^{(-m)*})$ may be approximated equiuniformly by step functions. By Corollary 8.1 and Proposition 3.4 we arrive at

$$\operatorname{ess\,sup}_{s \in [\tau, t]} \|\mathbb{E}_0(F_s^{(n)*} F_s^{(n)}) T_s \mathbb{E}_0(G_s^{(-m)} G_s^{(-m)*})\| \neq 0. \quad \blacksquare$$

In order to proceed, we need to know when time integrals are 0.

8.5. PROPOSITION. *Let $F, G \in \mathfrak{U}_1^{\tau}$ and $x, y \in \mathfrak{L}_{\operatorname{loc}}^{\infty}(E)$. Then $M_t = \int_{\tau}^t F \times d\mu^{x, y} G = 0$, if and only if*

$$\operatorname{ess\,sup}_{t \in [\tau, \infty)} \|F_t \langle x(t), y(t) \rangle G_t\| = 0.$$

Proof. By changing the function $F_t \langle x(t), y(t) \rangle G_t$ on a (measurable) null-set, we may achieve that $\operatorname{ess\,sup} \|\bullet\| = \sup \|\bullet\|$. Now the statement follows by Corollary 6.6. \blacksquare

8.6. LEMMA. *Let $F, G \in \mathfrak{U}_1^{\tau}$ and $x \in \mathfrak{L}_{\operatorname{loc}}^{\infty}(E)$. Then*

$$\operatorname{ess\,sup}_{t \in [\tau, \infty)} \|G_t^* \langle x(t), \mathbb{E}_0(F_t^* F_t) x(t) \rangle G_t\| = 0$$

$$\text{implies} \quad M_t = \int_{\tau}^t F d\ell^*(x) G = 0.$$

An analogue statement is true for annihilation integrals.

Proof. Of course, $\text{ess sup}_{t \in [\tau, \infty)} \|G_t^* \langle x(t), \mathbb{E}_0(F_t^* F_t) x(t) \rangle G_t\| = 0$ implies

$$\text{ess sup}_{t \in [\tau, \infty)} \|G_t^* \langle x(t), \mathbb{E}_0(F_t^{(n)*} F_t^{(n)}) x(t) \rangle G_t\| = 0$$

for all $n \in \mathbb{Z}$. By computations similar to (8.3) we find

$$\left| \int_{\tau}^t F^{(n)} d\ell^*(x) G \right|^2 = \int_{\tau}^t G^* d\mu^{x, \mathbb{E}_0(F_t^{(n)*} F_t^{(n)}) x} G$$

which is 0 by Proposition 8.5 so that $\int_{\tau}^t F^{(n)} d\ell^*(x) G = 0$ for all $n \in \mathbb{Z}$. Therefore, $M_t = \int_{\tau}^t F d\hat{\ell}^*(x) G = 0$. ■

8.7. Remark. The converse direction of Lemma 8.6 is done best by using the Ito formula. We postpone it to the following section. Notice, however, that computations like (8.3) already constitute an Ito formula in a particularly simple case.

9. ITO FORMULA

We start by introducing explicitly the notation which turns all integrals into conservation integrals, formally. For that goal, we consider the formal “operators” $\hat{\ell}^*(X)$ and $\hat{\ell}(X)$ where either $X = \omega$ (whence $\hat{\ell}^*(X) = \hat{\ell}(X) = \mathbf{1}$), or $X = x \in \mathfrak{L}_{\text{loc}}^{\infty}(E)$. This notation is formal in the sense that $\ell^*(x)$ and $\ell(x)$, in general, are not elements of $\mathcal{B}^a(\mathcal{F})$. In integrals they appear, however, only in combinations like $p(\chi_K) \ell^*(x) = \ell^*(\chi_K x)$ which are perfectly well-defined.

In this notation all integrals in Theorem 6.9 including the time integral can be written in the form

$$\int_{\tau}^{\mathcal{T}} F \hat{\ell}(X) dp(T) \hat{\ell}^*(Y) G$$

for suitable choices of X , Y , and T . By slight abuse of notation, we say $F \hat{\ell}(X) \in \mathfrak{U}_1^{\tau} \hat{\ell}(\mathcal{F}^{01})$ and $\hat{\ell}^*(X) G \in \hat{\ell}^*(\mathcal{F}^{01}) \mathfrak{U}_1^{\tau}$ where $\mathcal{F}^{01} = \mathbf{B} \oplus \mathfrak{L}_{\text{loc}}^{\infty}(E)$.

Of course, for creation, annihilation, or time integral we are reduced to $T = \mathbf{1}$. However, in the cases $X = x$, or $Y = y$, the operator $p(T)$ in $dp(T) = p(T) dp$ may be absorbed either into the creator on the right, or the annihilator on the left by Proposition 3.4.

9.1. THEOREM. Let M, M' be processes in \mathfrak{U}_1^τ given by integrals

$$M_t = \int_{\tau}^t F dp(T) G \quad \text{and} \quad M'_t = \int_{\tau}^t F' dp(T') G',$$

where $F, F' \in \mathfrak{U}_1^\tau \hat{\ell}(\mathcal{F}^{01})$, $G, G' \in \hat{\ell}^*(\mathcal{F}^{01}) \mathfrak{U}_1^\tau$, and $T, T' \in \mathfrak{L}^\infty(\mathbb{R}, \mathcal{B}^a(E))$. Then the product $MM' \in \mathfrak{U}_1^\tau$ is given by

$$M_t M'_t = \int_{\tau}^t F dp(T) G M'_t + \int_{\tau}^t M F' dp(T') G' + \int_{\tau}^t F dp(T \mathbb{E}_0(GF') T') G', \quad (9.1)$$

where $\mathbb{E}_0(GF')$ is the function $t \mapsto \mathbb{E}_0(G_t F'_t) \in \mathbf{B} \subset \mathcal{B}^a(\mathcal{F})$.

In differential notation $dM = F dp(T) G$ and $d(MM') = dMM' + M dM' + dM dM'$ we find the Ito formula

$$dM dM' = F dp(T \mathbb{E}_0(GF') T') G'.$$

Proof. Let us fix the compact interval $K = [\tau, \mathcal{T}]$. By Theorem 6.9 the nets $(F, p(T) G)_P$ and $(F' p(T'), G')_P$ converge *-strongly uniformly over $\mathcal{P}_{K'}$ for all compact intervalls $K' = [\tau, t] \subset K$ to M_t and M'_t , respectively. By Proposition 6.8 all nets are bounded uniformly for all $K' \subset K$. Therefore,

$$(F, p(T) G)_P (F' p(T'), G')_P \xrightarrow{* \text{-equistrongly}} M_t M'_t.$$

Splitting the double sum over k and ℓ into the parts where $k > \ell$, $k < \ell$, and $k = \ell$, we find

$$\begin{aligned} & (F, p(T) G)_P (F' p(T'), G')_P \\ &= \left[\sum_{1 \leq \ell < k \leq N} + \sum_{1 \leq k < \ell \leq N} \right] F_{t_{k-1}} dp_{t_k}(T) G_{t_{k-1}} F'_{t_{\ell-1}} dp_{t_{\ell}}(T') G'_{t_{\ell-1}} \\ &+ \sum_{k=1}^N F_{t_{k-1}} dp_{t_k}(T) G_{t_{k-1}} F'_{t_{k-1}} dp_{t_k}(T') G'_{t_{k-1}}. \end{aligned} \quad (9.2)$$

We will show that the first summand and the third summand of (9.2) converge strongly to the first summand and the third summand, respectively, of (9.1), establishing in this way that also the second summand of (9.2) converges strongly. Looking at the adjoint, we have formally the same sums, except that the first and the second summand have changed their roles. This shows that not only the limits are *-strong limits, but also that the limit of the second summand of (9.2) is the second summand of (9.1).

Let $Z \in \mathcal{F}_1$. By Theorem 6.9

$$\left\| \left(M'_{t_{k-1}} - \sum_{\ell=1}^{k-1} F'_{t_{\ell-1}} dp_{t_{\ell}}(T') G'_{t_{\ell-1}} \right) Z \right\|_1 < \varepsilon$$

for all k , if only the norm of $P \in \mathcal{P}_K$ is sufficiently small. Therefore, strong versions of (6.3a) and (6.3b) (depending on whether $G \in \mathfrak{U}_1^{\tau}$ or $G \in \ell(y) \mathfrak{U}_1^{\tau}$) tell us that the first summand in (9.2) converges strongly to the first summand in (9.1).

For the last summand of (9.1) we assume concretely that $F = \bar{F}\hat{\ell}(X)$ and $G = \hat{\ell}^*(Y) \bar{G}$ ($\bar{F}, \bar{G} \in \mathfrak{U}_1$; $X, Y \in \mathcal{F}^{01}$), and similarly for F', G' . For the case $Y = X' = \omega$ we find from Corollary 4.6 and the proof of Theorem 6.9 convergence in norm. In the remaining cases $\mathbb{E}_0(G_t F'_t)$ is 0. Let us check whether this is also true for the limit of the last summand of (9.2). For instance, assume that $Y = y \in \mathfrak{L}_{\text{loc}}^{\infty}(E)$. We find

$$\begin{aligned} & \left\| \sum_{k=1}^N F_{t_{k-1}} dp_{t_k}(T) G_{t_{k-1}} F'_{t_{k-1}} dp_{t_k}(T') G'_{t_{k-1}} \right\| \\ & \leq \|(F, p(T))_P\| \left\| \sum_{k=1}^N dp_{t_k} G_{t_{k-1}} F'_{t_{k-1}} dp_{t_k} \right\| \|(p(T'), G')_P\|. \end{aligned}$$

For the square modulus of the sum we find by computations as in Lemma 6.7

$$\begin{aligned} & \sum_{k=1}^N dp_{t_k} F'^*_{t_{k-1}} G^*_{t_{k-1}} dp_{t_k} G_{t_{k-1}} F'_{t_{k-1}} dp_{t_k} \\ & \leq \sum_{k=1}^N \|\langle y, dp_{t_k} \rangle\| \|F'^*_{t_{k-1}} \bar{G}^*_{t_{k-1}} \bar{G}_{t_{k-1}} F'_{t_{k-1}}\| dp_{t_k} \\ & \leq \max_{1 \leq k \leq N} (\|\langle y, dp_{t_k} \rangle\|) \|F'^* \bar{G}^* \bar{G} F'\|. \end{aligned}$$

As the first factor tends to 0, we find convergence to 0 also in norm. ■

9.2. COROLLARY. *Let $M_t = \int_0^t F dI G$ and $M'_t = \int_0^t F' dI' G'$ be integrals as in Theorem 6.9. Then $dM dM' = F dI'' G'$ where dI'' has to be chosen according to the Ito table*

$dI \backslash dI'$	$d\mu^{x', y'}$	$d\ell^*(x')$	$d\ell(x')$	$dp(T')$
$d\mu^{x, y}$	0	0	0	0
$d\ell^*(x)$	0	0	0	0
$d\ell(x)$	0	$d\mu^{x, \mathbb{E}_0(GF') x'}$	0	$d\ell(T'^* \mathbb{E}_0(GF')^* x)$
$dp(T)$	0	$d\ell^*(T \mathbb{E}_0(GF') x')$	0	$dp(T \mathbb{E}_0(GF') T')$

9.3. *Remark.* It is easy to see that the Ito formula extends also to more general time integrals $\int F dI$ where I is an integrator with a locally bounded density $I' \in \mathfrak{L}_{\text{loc}}^\infty(\mathbf{B})$. Of course, also Proposition 8.5 remains true replacing $\langle x(t), y(t) \rangle$ with a more general density I' .

9.4. **THEOREM.** *Let $F^i, G^i \in \mathfrak{U}_1^\tau$ ($i=0, +, -, 1$), $x, y \in \mathfrak{L}_{\text{loc}}^\infty(E)$, $T \in \mathfrak{L}_{\text{loc}}^\infty(\mathcal{B}^a(E))$, and let I be an integrator with locally bounded density $I' \in \mathfrak{L}_{\text{loc}}^\infty(\mathbf{B})$. Let*

$$M_t = \int_\tau^t dM^0 + \int_\tau^t dM^+ + \int_\tau^t dM^- + \int_\tau^t dM^1$$

be a sum of integrals where $dM^0 = F^0 dI G^0$, $dM^+ = F^+ d\ell^(x) G^+$, $dM^- = F^- d\ell(y) G^-$, and $dM^1 = F^1 dp(T) G^1$. Then the following conditions are equivalent.*

$$(1) \quad M = 0.$$

$$(2) \quad \int dM^0 = \int dM^+ = \int dM^- = \int dM^1 = 0.$$

$$(3) \quad \text{ess sup}_{t \in [\tau, \infty)} \|F_t^0 I'_t G_t^0\| = 0,$$

$$\text{ess sup}_{t \in [\tau, \infty)} \|\mathbb{E}_0(F_t^1 * F_t^1) T_t \mathbb{E}_0(G_t^1 G_t^1 *)\| = 0$$

$$\text{ess sup}_{t \in [\tau, \infty)} \|G_t^{+*} \langle x(t), \mathbb{E}_0(F_t^+ * F_t^+) x(t) \rangle G_t^+\| = 0$$

$$\text{ess sup}_{t \in [\tau, \infty)} \|F_t^- \langle y(t), \mathbb{E}_0(G_t^- G_t^- *) y(t) \rangle F_t^{-*}\| = 0.$$

Proof. By Proposition 8.5 and Lemmata 8.4 and 8.6 we have (3) \Rightarrow (2) and, of course, we have (2) \Rightarrow (1).

So let us assume $M = 0$. In particular, M is continuous. Since $\int dM^0 + \int dM^+ + \int dM^-$ is continuous by Proposition 6.8, so is $\int dM^1$. By Lemma 8.4 we conclude that $\int dM^1 = 0$, and that the condition in (3) concerning the conservation integral is fulfilled.

Writing down the Ito formulae for $M^* M$ and $M M^*$, and taking into account that $M = M^* = 0$ and that the conservation part is absent, we find that $\int dM^* dM = \int G^{+*} d\mu^x, \mathbb{E}_0(F^{+*} F^+) x G^+ = 0$ and $\int dM dM^* = \int F^- d\mu^y, \mathbb{E}_0(G^- G^{-*}) y F^{-*} = 0$. Therefore, by Proposition 8.5 also the conditions in (3) concerning creation and annihilation part must be fulfilled.

Since all parts except the time integral are known to be 0, also the time integral must be 0. Again by Proposition 8.5 we find that also the last condition in (3) must be fulfilled. This is (1) \Rightarrow (3). ■

10. UNITARITY CONDITIONS

We are interested in finding necessary and sufficient conditions under which a solution U of a differential equation like (7.2) is unitary. Usually, this is done by writing down what the Ito formula asserts for

$$d(U^*U) = dU^* U + U^* dU + dU^* dU \quad (10.1a)$$

and

$$d(UU^*) = dU U^* + U dU^* + dU dU^*. \quad (10.1b)$$

If the coefficients of all summands in these expressions are 0, then this is certainly sufficient to conclude that U is unitary. To have necessity we must conclude backwards from $\int d(U^*U) = \int d(UU^*) = 0$ that also all coefficients vanish. Presently, however, we have only the criterion Theorem 9.4, where each type of integrators dl , $d\ell^*$, $d\ell$, dp appears not more than once. Unfortunately, even in differential equations of the simpler form (7.1) the Ito formula yields, in general, more summands of the same type which cannot be summed up to a single one.

Here we consider differential equations *without coefficients*. This means that there are no processes F, G around the integrators. At first sight, this looks poor. However, we allow for rather arbitrary arguments in the integrators. As we explain in Section 13, this is already sufficient to include the case of a calculus on a full Fock space with initial space and arbitrarily many degrees of freedom. (In [KS92, Spe91] only the Fock space over $L^2(\mathbb{R})$ is considered which, roughly speaking, corresponds to one degree of freedom. In the unitarity conditions in [Spe91] at least some of the processes around the integrators may vary over \mathfrak{U}_1 . So, at least in the cases where [Spe91] applies the conditions given there are more general.) The proof of the following theorem is very much along the lines of the corresponding proof in [Spe91].

10.1. THEOREM. *Let $x, y \in \mathfrak{L}_{\text{loc}}^\infty(E)$, $T \in \mathfrak{L}_{\text{loc}}^\infty(\mathcal{B}^a(E))$, and let l be an integrator with locally bounded density $l' \in \mathfrak{L}_{\text{loc}}^\infty(\mathbf{B})$. Then the unique solution in \mathfrak{U}_1^0 of the differential equation*

$$dU = U(dp(T) + d\ell^*(x) + d\ell(y) + dl), \quad U_0 = \mathbf{1} \quad (10.2)$$

is unitary, if and only if the following conditions are fulfilled.

- (1) $T(t) + \mathbf{1}$ is unitary almost everywhere on \mathbb{R}^+ .
- (2) $x(t) + T(t) y(t) + y(t) = 0$ almost everywhere on \mathbb{R}^+ .
- (3) $l'(t) + l'^*(t) + \langle x(t), x(t) \rangle = 0$ almost everywhere on \mathbb{R}^+ .

Proof. From

$$dU^* = (dp(T^*) + d\ell^*(y) + d\ell(x) + dl^*) U^*, \quad U_0^* = \mathbf{1}$$

we find for (10.1a), (10.1b) the explicit expressions

$$\begin{aligned} d(U^*U) &= (dp(T^*) + d\ell^*(y) + d\ell(x) + dl^*) U^*U \\ &\quad + U^*U(dp(T) + d\ell^*(x) + d\ell(y) + dl) \\ &\quad + dp(T^*\mathbb{E}_0(U^*U)T) + d\ell^*(T^*\mathbb{E}_0(U^*U)x) \\ &\quad + d\ell(T^*\mathbb{E}_0(U^*U)x) + d\mu^{x, \mathbb{E}_0(U^*U)x} \end{aligned} \quad (10.3a)$$

and

$$\begin{aligned} d(UU^*) &= U(dp(T) + d\ell^*(x) + d\ell(y) + dl) \\ &\quad + dp(T^*) + d\ell^*(y) + d\ell(x) + dl^* \\ &\quad + dp(TT^*) + d\ell^*(Ty) + d\ell(Ty) + d\mu^{y, y} U^* \\ &= U(dp(T + T^* + TT^*) + d\ell^*(x + y + Ty) + d\ell(x + y + Ty) \\ &\quad + (dl + dl^* + d\mu^{y, y})) U^*. \end{aligned} \quad (10.3b)$$

If U is unitary, then $\mathbb{E}_0(U^*U) = \mathbf{1}$ and (10.3a) simplifies to

$$\begin{aligned} 0 &= dp(T + T^* + T^*T) + d\ell^*(x + y + T^*x) + d\ell(x + y + T^*x) \\ &\quad + (dl + dl^* + d\mu^{x, x}). \end{aligned}$$

By Theorem 9.4 we find $(T + T^* + T^*T)(t) = 0$ (i.e., $(T + \mathbf{1})(t)$ is an isometry), $(x + y + T^*x)(t) = 0$, and $l(t) + l^*(t) + \langle x(t), x(t) \rangle = 0$ for almost all $t \in \mathbb{R}^+$. Equation (10.3b) implies (notice that U and U^* disappear in all suprema in Theorem 9.4, if U is unitary) that also $(T + T^* + TT^*)(t) = 0$ for almost all $t \in \mathbb{R}^+$. In other words, $(T + \mathbf{1})(t)$ is a unitary, such that also $(x + y + Ty)(t) = (T + \mathbf{1})(x + y + T^*x)(t) = 0$ and $d\mu^{y, y} = d\mu^{x, x}$.

Conversely, if the three conditions are fulfilled, then by (10.3b), $d(UU^*) = 0$. Together with the initial condition $(UU^*)_0 = \mathbf{1}$ we find that U is a coisometry. Whereas, U^*U fulfills the differential Eq. (10.3a) also with initial condition $(U^*U)_0 = \mathbf{1}$. One easily checks that $U^*U = \mathbf{1}$ is a solution of (10.3a). By Theorem 7.3 this solution is unique. Therefore, U is unitary. ■

It is noteworthy that, although our differential equation has no coefficients F and G , we needed Lemma 8.4 in full generality in order to be able to conclude from (10.3b) to $T + T^* + TT^* = 0$.

A more common way to write down a differential equation with unitary solution is

$$dU = U(dp(W - \mathbf{1}) + d\ell^*(Wy) - d\ell(y) + (i dH - \frac{1}{2} d\mu^{y,y})), \quad U_0 = \mathbf{1},$$

where W is unitary, y is arbitrary, and H is self-adjoint.

11. COCYCLES

Let us return for a moment to the differential equation in the form (10.2) (without unitarity conditions). U_t is adapted to E_t and the differentials dp , $d\ell^*$, $d\ell$, dI are adapted to the complement of E_t . As pointed out in [Spe98] this means that in the sense of Voiculescu [Voi95] U_t and the differentials are *freely independent with amalgamation over \mathbf{B}* in the vacuum conditional expectation. In other words, U is a process with independent (right) multiplicative increments.

If we choose constant functions $T(t) = t$, $x(t) = \xi$, $y(t) = \zeta$, and $I'(t) = j$ (with $t \in \mathcal{B}^a(E)$, $\xi, \zeta \in E$, $j \in \mathbf{B}$), then U has even *stationary* increments. The goal of this section is to show that in this case U is a cocycle with respect to the time shift automorphism group on $\mathcal{B}^a(\mathcal{F})$. The results by Hellmich, Köstler, and Kümmerer [HKK98] indicate that (at least, when \mathbf{B} is a von Neumann algebra with a faithful normal state) for unitary cocycles U also the converse is true.

In the sequel, we identify a constant function in some \mathfrak{L}^∞ -space with its constant value. It should be clear from the context whether we refer to the constant function or its value.

11.1. DEFINITION. We define the *time shift* ϑ_t ($t \in \mathbb{R}$) on the one-particle sector E_∞ by setting $[\vartheta_t x](s) = x(s - t)$. Obviously, ϑ_t is \mathbf{B} - \mathbf{B} -linear and unitary. The *time shift* on \mathcal{F} is the second quantization $\mathcal{F}(\vartheta_t)$; cf. Definition 3.3. Also $\mathcal{F}(\vartheta_t)$ is \mathbf{B} - \mathbf{B} -linear and unitary.

The *time shift automorphism group* $\mathcal{S} = (\mathcal{S}_t)_{t \in \mathbb{R}}$ on $\mathcal{B}^a(\mathcal{F})$ is defined by setting $\mathcal{S}_t(a) = \mathcal{F}(\vartheta_t) a \mathcal{F}(\vartheta_t)^*$. As $\mathcal{F}(\vartheta_t)$ is bilinear, \mathcal{S}_t leaves invariant $\mathbf{B} \subset \mathcal{B}^a(\mathcal{F})$.

A process $U = (U_t)_{t \in \mathbb{R}}$ of operators on \mathcal{F} is a *left cocycle* (with respect to \mathcal{S}), if $U_{s+t} = U_s \mathcal{S}_s(U_t)$ ($s, t \geq 0$). If U is a unitary left cocycle, then one easily checks that $\mathcal{S}^U = (\mathcal{S}_t^U)_{t \in \mathbb{R}^+}$ defined by setting $\mathcal{S}_t^U(a) = U_t \mathcal{S}_t(a) U_t^*$ is an automorphism semigroup.

The proof of the following theorem is like in [KS92]. We just do not require that the cocycle be unitary.

11.2. THEOREM. Let $t \in \mathcal{B}^a(E)$, $\xi, \zeta \in E$, and $J \in \mathbf{B}$. Then the solution of

$$dU = U(dp(t) + d\ell^*(\xi) + d\ell(\zeta) + J dt), \quad U_0 = \mathbf{1} \quad (11.1)$$

is an adapted left cocycle.

Proof. Thanks to the stationarity of the differential (i.e., the arguments of the integrators do not depend on time) we have the substitution rule

$$\begin{aligned} \mathcal{S}_s \left(\int_0^t F_{t'}(dp_{t'}(t) + d\ell_{t'}^*(\xi) + d\ell_{t'}(\zeta) + J dt') G_{t'} \right) \\ = \int_s^{s+t} \mathcal{S}_s(F_{t'-s})(dp_{t'}(t) + d\ell_{t'}^*(\xi) + d\ell_{t'}(\zeta) + J dt') \mathcal{S}_s(G_{t'-s}) \end{aligned}$$

which is easily verified by looking at the definitions of the integrals in Theorem 6.9. We insert this for U_t and find

$$U_s \mathcal{S}_s(U_t) = U_s + \int_s^{s+t} U_s \mathcal{S}_s(U_{t'-s})(dp_{t'}(t) + d\ell_{t'}^*(\xi) + d\ell_{t'}(\zeta) + J dt'). \quad (11.2)$$

In other words, the process $U'_t = U_s \mathcal{S}_s(U_{t-s})$ fulfills for $t \in [s, \infty)$ the same differential equation as U_t with the same initial condition $U'_s = U_s$, i.e., $U_t = U'_t$ for $t \geq s$. ■

Notice that the initial condition $U_0 = \mathbf{1}$ (or at least a condition like $U_s \mathcal{S}_s(U_0) = U_s$ for all s) is indispensable. Otherwise, the first summand in (11.2) was $U_s \mathcal{S}_s(U_0)$ so that we gain the wrong initial value.

12. DILATIONS

A *CP-semigroup* on a C^* -algebra \mathbf{B} is a semigroup $T = (T_t)_{t \in \mathbb{R}^+}$ of completely positive mappings $T_t: \mathbf{B} \rightarrow \mathbf{B}$, i.e., $T_0 = \text{id}$, $T_s \circ T_t = T_{s+t}$, and

$$\sum_{i,j} b_i^* T_t(a_i^* a_j) b_j \geq 0$$

for all choices of finitely many $a_i, b_i \in \mathbf{B}$. A CP-semigroup T on a unital C^* -algebra is called *conservative*, if $T_t(\mathbf{1}) = \mathbf{1}$ ($t \in \mathbb{R}^+$).

In general, the goal of dilation theory is to embed \mathbf{B} into a bigger C^* -algebra \mathbf{A} in such a way that there exists a *conditional expectation* $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ (i.e., a norm-one projection onto \mathbf{B}) and a semigroup $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$ of endomorphisms of \mathbf{A} , fulfilling $T_t = \varphi \circ \mathcal{G}_t$. These are the weakest requirements a dilation should fulfill. There are, however, several ways of strengthening the notion of dilation. First, we can require that \mathcal{G}

consists of unital endomorphisms or even automorphisms. In the latter case \mathcal{G} extends to an automorphism group. Second, we can require that \mathbf{B} is embedded unittally, i.e., $\mathbf{1}_{\mathbf{B}} = \mathbf{1}_{\mathbf{A}}$.

In this section we construct dilations of uniformly continuous CP-semigroups where \mathbf{A} is the algebra of adjointable operators on a suitably chosen Fock module $\mathcal{F} = \mathcal{F}(E_{\infty})$. Then \mathbf{B} is embedded unittally into \mathbf{A} and the conditional expectation is just the vacuum conditional expectation \mathbb{E}_0 . We find these dilations by *perturbing* the time shift \mathcal{S} (which leaves invariant \mathbf{B}) by an adapted unitary cocycle U in the sense of Definition 11.1. In other words, we dilate T to the automorphism semigroup \mathcal{S}^U (which, of course, may be extended to an automorphism group). As usual, U is the solution of a differential equation. In Remark 12.3 we point out how we can obtain a dilation to an E_0 -semigroup (i.e., a semigroup of unital endomorphisms of \mathbf{A} rather than an automorphism group) by restricting the one-particle sector to times $t \geq 0$. This is more similar to the approach in [HP84].

We construct the one-particle sector of our Fock module from the generator of T . Let $T = (T_t)_{t \in \mathbb{R}^+}$ be a conservative CP-semigroup on a unital C^* -algebra \mathbf{B} which is uniformly continuous or, equivalently, which has a bounded *generator*, i.e., a bounded linear mapping $L: \mathbf{B} \rightarrow \mathbf{B}$ such that $T_t = e^{tL}$. Christensen and Evans [CE79] show that the generator has the form

$$L(b) = L^0(b) - \frac{bL^0(\mathbf{1}) + L^0(\mathbf{1})b}{2} + i[h, b], \quad (12.1)$$

where $L^0: \mathbf{B} \rightarrow \mathbf{B}^{**}$ is a completely positive mapping (in general, neither unital nor contractive) and $h \in \mathbf{B}^{**}$ is self-adjoint. It is far from being clear under which circumstances L^0 takes values in \mathbf{B} and $h \in \mathbf{B}$. We only know that the sum of all elements in \mathbf{B}^{**} appearing (12.1) is an element of \mathbf{B} . However, as pointed out in [CE79] the extension T^{**} of T to the bidual \mathbf{B}^{**} is a CP-semigroup with generator L^{**} of the same form (12.1) where we just have to extend L^0 to \mathbf{B}^{**} . Therefore, we always may assume that (possibly after extension to the bidual) L^0 leaves invariant \mathbf{B} and that $h \in \mathbf{B}$. In this case we speak of a generator of Christensen–Evans type.

Once, we have the completely positive mapping L^0 , we may do the GNS-construction; see Paschke [Pas73] or [Ske97, BS99] for details. In other words, we define a \mathbf{B} -valued semi-inner product on $\mathbf{B} \otimes \mathbf{B}$ (with its obvious \mathbf{B} – \mathbf{B} -module structure) by setting

$$\langle a \otimes b, a' \otimes b' \rangle = b^* L^0(a^* a') b'.$$

By dividing out the submobule \mathcal{N} of *length-zero elements* and completion we obtain a Hilbert \mathbf{B} – \mathbf{B} -module E and a *cyclic vector* $\xi = \mathbf{1} \otimes \mathbf{1} + \mathcal{N}$ such

that $L^0(b) = \langle \xi, b\xi \rangle$. We refer to the pair (E, ξ) as the *GNS-construction* of L^0 and we refer to E as the *GNS-module*.

12.1. THEOREM. *Let $T = (T_t)_{t \in \mathbb{R}^+}$ be a conservative CP-semi on a unital C^* -algebra \mathbf{B} with bounded generator L of Christensen–Evans type, i.e., L has the form (12.1) for some completely positive mapping L^0 on \mathbf{B} and some self-adjoint element $h \in \mathbf{B}$.*

Denote by (E, ξ) the GNS-construction for L^0 so that

$$L(b) = \langle \xi, b\xi \rangle - \frac{b \langle \xi, \xi \rangle + \langle \xi, \xi \rangle b}{2} + i[h, b]. \quad (12.2)$$

Let w be a unitary in $\mathcal{B}^a(E)$. Let U be the adapted unitary left cocycle obtained as the unique solution of the differential equation

$$dU = U(dp(w - \mathbf{1}) + d\ell^*(w\xi) - d\ell(\xi) + (ih - \tfrac{1}{2}\langle \xi, \xi \rangle) dt), \quad U_0 = \mathbf{1}. \quad (12.3)$$

Then \mathcal{S}^U is a dilation of T , i.e., $T_t = \mathbb{E}_0 \circ \mathcal{S}_t^U \upharpoonright \mathbf{B}$.

Conversely, if E is a Hilbert \mathbf{B} – \mathbf{B} -module and $\xi \in E$, $h \in \mathbf{B}$, then by setting $T_t = \mathbb{E}_0 \circ \mathcal{S}_t^U \upharpoonright \mathbf{B}$, where U is the adapted unitary left cocycle fulfilling (12.3), we define a uniformly continuous conservative CP-semigroup T whose generator L is given by (12.2).

Proof. It is enough to show that for U given by (12.3) the family $T_t(b) = \mathbb{E}_0 \circ \mathcal{S}_t^U(b)$ fulfills $T'_t(b) = T_t \circ L(b)$.

As \mathcal{S} leaves invariant \mathbf{B} , we have $\mathcal{S}_t^U(b) = U_t b U_t^*$. Applying, for fixed $b \in \mathbf{B}$, the Ito formula to this product of integrals, we find

$$\begin{aligned} d\mathcal{S}^U(b) &= dU b U^* + U b dU^* + dU b dU^* \\ &= (U((dp(w - \mathbf{1}) + d\ell^*(w\xi) - d\ell(\xi) + (ih - \tfrac{1}{2}\langle \xi, \xi \rangle) dt) b \\ &\quad + b(dp(w^* - \mathbf{1}) - d\ell^*(\xi) + d\ell(w\xi) - (ih + \tfrac{1}{2}\langle \xi, \xi \rangle) dt) \\ &\quad + dp((w - \mathbf{1}) b (w^* - \mathbf{1})) - d\ell^*((w - \mathbf{1}) b \xi) - d\ell((w - \mathbf{1}) b^* \xi) \\ &\quad + \langle \xi, b\xi \rangle dt) U^*. \end{aligned}$$

By Lemma 4.5 in all summands containing dp or $d\ell^*$ we may replace U on the left by $\hat{\ell}^*(U\omega)$ and in all summands containing dp or $d\ell$ we may replace U^* on the right by $\hat{\ell}(U\omega)$. It follows that applying the vacuum conditional expectation only the time differentials survive. As $\mathbb{E}_0: \mathcal{B}^a(\mathcal{F}) \rightarrow \mathbf{B}$ is continuous in the $*$ -strong topology on $\mathcal{B}^a(\mathcal{F})$ and the uniform topology on \mathbf{B} , it follows that

$$\begin{aligned}
T_t(b) - b &= \int_0^t \mathbb{E}_0(U_s(((ih - \tfrac{1}{2} \langle \xi, \xi \rangle) b - b(ih + \tfrac{1}{2} \langle \xi, \xi \rangle) \\
&\quad + \langle \xi, b\xi \rangle) ds) U_s^*) \\
&= \int_0^t \mathbb{E}_0(U_s \mathbb{L}(b) U_s^* ds) = \int_0^t T_s \circ \mathbb{L}(b) ds. \quad \blacksquare
\end{aligned}$$

12.2. *Remark.* As usual with dilations obtained by a calculus including conservation integrals, we obtain a whole family of dilations classified by the unitary w . However, notice that the dilation corresponding to $w = \mathbf{1}$ plays a distinguished role. In this case the conservation integral disappears and the cocycle U is continuous. We already mentioned that in the case without conservation integral we may restrict the set of processes under consideration to continuous processes and that everything converges in norm. On the other hand, if $w \neq \mathbf{1}$ we know by Lemma 8.4 that U is certainly not continuous.

12.3. *Remark.* It is easily possible to restrict the situation to operators on the Fock module $\mathcal{F}_{\mathbb{R}^+} = \mathcal{F}(L^2([0, \infty), E))$. In fact, $\mathcal{B}^a(\mathcal{F}_{\mathbb{R}^+})$ may be indentified with those operators in $\mathcal{B}^a(\mathcal{F})$ which are adapted to $L^2([0, \infty), E)$. Everything concerning integrals, Ito formula, differential equations, etc. goes through, if we restrict to processes in \mathfrak{U}_1 with values in $\mathcal{B}^a(\mathcal{F}_{\mathbb{R}^+})$. One easily checks that the time shift automorphism \mathcal{S}_t leaves invariant $\mathcal{B}^a(\mathcal{F}_{\mathbb{R}^+})$ for all $t \geq 0$. Of course, the restriction of \mathcal{S}_t to $\mathcal{B}^a(\mathcal{F}_{\mathbb{R}^+})$ is no longer an automorphism, but a unital endomorphism. It still remains true that U is a unitary left cocycle with respect to the restriction of \mathcal{S} . Summing up we obtain a dilation of T to the E_0 -semigroup $\mathcal{S}^U \upharpoonright \mathcal{B}^a(\mathcal{F}_{\mathbb{R}^+})$. It is noteworthy that our limits are in the $*$ -strong topology of $\mathcal{B}^a(\mathcal{F})$ which, clearly, is stronger than the original $*$ -strong topology of $\mathcal{B}^a(\mathcal{F}_{\mathbb{R}^+})$ (not considered as a subset of $\mathcal{B}^a(\mathcal{F})$).

13. THE CASE $\mathbf{B} = \mathcal{B}(G)$

Suppose $\mathbf{B} \subset \mathcal{B}(G)$ is a concrete unital C^* -algebra of operators on a Hilbert space G . Then the tensor product $E \odot G$ of a Hilbert \mathbf{B} -module E and the Hilbert \mathbf{B} - \mathbb{C} -module G is a Hilbert \mathbb{C} -module, i.e., a Hilbert space. $\mathcal{B}(G, E \odot G)$ is a Hilbert $\mathcal{B}(G)$ -module with inner product $\langle L, L' \rangle = L^* L'$ and obvious module structure.

For $x \in E$ we define the mapping $L_x \in \mathcal{B}(G, E \odot G)$ by setting $L_x g = x \odot g$. Then $\langle x, x' \rangle = L_x^* L_{x'}$. Henceforth, we identify E as a subset of $\mathcal{B}(G, E \odot G)$ and do no longer distinguish between $x \in E$ and $L_x \in \mathcal{B}(G, E \odot G)$.

If A is another C^* -algebra and E is a Hilbert A – B -module (in particular, each Hilbert B -module is also a Hilbert $\mathcal{B}^a(E) - B$ -module in the obvious way), then we define a representation $\rho: A \rightarrow \mathcal{B}(E \odot G)$ by setting $\rho(a)(x \odot g) = ax \odot g$. The norm of a as an operator on E and $\|\rho(a)\|$ coincide. Usually, we identify also a and $\rho(a)$.

If B is a von Neumann algebra on G , then we say E is a *von Neumann B -module*, if it is a strongly closed subset of $\mathcal{B}(G, E \odot G)$; see [Ske97, BS99] for details. Let A be another von Neumann algebra. If E is a von Neumann B -module and a Hilbert A – B -module, and if the representation of A on $E \odot G$ is normal, then we say E is a *von Neumann A – B -module*; see [Ske97].

For us it is important to know that the strong closure of the GNS-module of a normal completely positive mapping $T: A \rightarrow B$ is a von Neumann A – B -module, and that the strong closure of the tensor product of two von Neumann modules is again a von Neumann module. All these assertions are shown in the appendix of [BS99]. It follows that, starting from a normal conservative CP-semigroup T with (σ -weak!) bounded generator L , all our operators in the calculus extend to the strong closure of the Fock module (and it plays no role, if we close the one-particle sector first) and we find a dilation of T on this closure.

Now since we know that our results extend to von Neumann modules, we restrict our attention to the special case $B = \mathcal{B}(G)$. We collect some results from [BS99]. If E is a von Neumann $\mathcal{B}(G)$ -module, then $E = \mathcal{B}(G, E \odot G)$. (To see this, observe that $\mathcal{B}(G)$ contains all rank-one operators, hence, so does E .) In this case $\mathcal{B}^a(E)$ coincides with $\mathcal{B}(E \odot G)$ via ρ ; see [Ske98a]. If E is a von Neumann $\mathcal{B}(G)$ – $\mathcal{B}(G)$ -module, then $E = \mathcal{B}(G, G \otimes H)$ with its obvious $\mathcal{B}(G)$ – $\mathcal{B}(G)$ -module structure. ($E \odot G$ carries a non-degenerate normal representation of $\mathcal{B}(G)$. Therefore, this representation is unitarily equivalent to the representation $\text{id} \otimes \mathbf{1}$ on $G \otimes H$ where H is a suitable Hilbert space.)

The von Neumann module $\mathcal{B}(G, G \otimes H)$ is particularly simple example of what we call a *centered* module in [Ske98a] (i.e., $\mathcal{B}(G, G \otimes H)$ is topologically generated as a right module by those elements which commute with all elements of $\mathcal{B}(G)$). Observe that $\mathcal{B}(G, G \otimes H)$ has a module basis. Indeed, let us denote by $b \otimes f$ ($b \in \mathcal{B}(G)$, $f \in H$) the mapping $g \mapsto bg \otimes f$ in $\mathcal{B}(G, G \otimes H)$. Then for any orthonormal basis $(e_i)_{i \in \mathbf{I}}$ of H the mappings $\mathbf{1} \otimes e_i$ form an “orthonormal basis” for $\mathcal{B}(G, G \otimes H)$ in the sense that each element $\xi \in \mathcal{B}(G, G \otimes H)$ can be expressed as strongly convergent sum

$$\xi = \sum_{i \in \mathbf{I}} b_i \otimes e_i$$

with unique coefficients $b_i = \langle \mathbf{1} \otimes e_i, \xi \rangle \in \mathcal{B}(G)$.

Let us apply this to the cyclic vector of the GNS-construction for the completely positive part L^0 of a generator L of a normal conservative CP-semigroup T on $\mathcal{B}(G)$. We recover the well known Lindblad form

$$L(b) = \sum_{i \in I} b_i^* b b_i - \frac{b \sum_{i \in I} b_i^* b_i + \sum_{i \in I} b_i^* b_i b}{2} + i[h, b] \quad (13.1)$$

of the generator [Lin76]. Also the unitary operator w appearing in the differential equation (12.3) can be expanded according to the basis. We find a matrix $(b_{ij})_{i, j \in I}$ of elements in $\mathcal{B}(G)$ such that $w(\mathbf{1} \otimes e_i) = \sum_{j \in I} b_{ij} \otimes e_j$. Expressing all ingredients of (12.3) in this way, we find an expansion of our integrators into “basic integrators” $dp(|e_i\rangle\langle e_j|)$, $d\ell^*(e_i)$, and $d\ell(e_i)$ as used in [MS90, Par92] in the *calculus on the symmetric Fock space with arbitrary degree of freedom*. The Mohari–Sinha regularity conditions

$$\sum_{i \in I} b_i^* b_i < \infty \quad \text{and} \quad \sum_{i \in I} b_{ij}^* b_{ij} < \infty \quad \text{for all } j \in I$$

mean just that ξ is a well defined in $\mathcal{B}(G, G \otimes H)$ and that w is a well-defined operator at least on the $\mathcal{B}(G)$ -linear span of all $\mathbf{1} \otimes e_i$. If the constant in the above condition for b_{ij} does not depend on j then w is a bounded operator on $\mathcal{B}(G, G \otimes H)$.

Notice that the strong closure of $L^2(\mathbb{R}, \mathcal{B}(G, G \otimes H))$ is $\mathcal{B}(G, G \otimes L^2(\mathbb{R}) \otimes H)$. The Fock module is nothing but $\mathcal{F} = \mathcal{B}(G, G \otimes \mathcal{F}(L^2(\mathbb{R}) \otimes H))$ where $\mathcal{F}(L^2(\mathbb{R}) \otimes H)$ is the usual Fock space over the Hilbert space $L^2(\mathbb{R}) \otimes H$. Finally, we find $\mathcal{F} \odot G = G \otimes \mathcal{F}(L^2(\mathbb{R}) \otimes H)$. Since the operator algebras $\mathcal{B}^a(\mathcal{F})$ and $\mathcal{B}(\mathcal{F} \odot G)$ coincide, our calculus can be interpreted as a calculus with arbitrary (even uncountable) degree of freedom on the tensor product of the *initial space* G and the full Fock space $\mathcal{F}(L^2(\mathbb{R}) \otimes H)$. In [Spe91] only the case $H = \mathbb{C}$ is treated, which corresponds to one degree of freedom.

Let us summarize. Although we follow in many respects directly the ideas in [KS92, Spe91], we can say that our calculus is both formally simpler and more general. It is formally simpler, because our differential equation for U contains no coefficients. (Of course, the coefficients are hidden in the much more general arguments of the integrators.) And our calculus is more general, because it allows to find dilations for arbitrary Christensen–Evans generators. As a special case we showed in this section how the calculus for an arbitrary Lindblad generator is contained, which on a Fock space—symmetric or full—requires a calculus with arbitrary degree of freedom.

Recently, in [GS99] a calculus on the *symmetric Fock* $\mathcal{B}(G)$ – $\mathcal{B}(G)$ -module $\mathcal{B}(G, G \otimes \Gamma(L^2(\mathbb{R}^+) \otimes H))$ has been constructed. This von Neumann module can, indeed, be considered as the strong closure of the symmetric Fock

module over $L^2(\mathbb{R}^+, \mathcal{B}(G) \otimes H)$ as defined for arbitrary centered Hilbert modules in [Ske98a]. This calculus allowed for the first time to dilate an arbitrary generator of Christensen–Evans type (and also the construction of Evans–Hudson flows, which we do not consider at all). The construction of the one-particle sector in [GS99] is, however, less canonical in the following sense. The completely positive part L^0 of the generator L gives rise only to a \mathbf{B} – \mathbf{B} -module. Before finding the $\mathcal{B}(G)$ – $\mathcal{B}(G)$ -module $\mathcal{B}(G) \otimes H$, from which the symmetric Fock module can be constructed, it is necessary to extend the module structure from \mathbf{B} (which is rarely centered) to $\mathcal{B}(G)$ (which is always centered). Also the techniques in [GS99] refer more to Hilbert spaces, which do not play a role in our treatment.

14. BOOLEAN CALCULUS

There are several possibilities to translate the concept of *independence* from classical (or commutative) probability to quantum (or non-commutative) probability. The minimal requirement for a notion of non-commutative independence is probably that used by Kümmerer [Küm85], where (speaking about unital $*$ -algebras and states instead of von Neumann algebras and faithful normal states) two (unital) $*$ -subalgebras \mathbf{A}_i ($i = 1, 2$) of a $*$ -algebra \mathbf{A} are independent in a state φ on \mathbf{A} , if $\varphi(a_j a_k) = \varphi(a_j) \varphi(a_k)$ ($k \neq j$; $a_i \in \mathbf{A}_i$; $i = 1, 2$).

A more specific notion of non-commutative independence as introduced in [Sch95] requires that the values of φ on alternating monomials in \mathbf{A}_1 and \mathbf{A}_2 may be obtained from a *universal product* $\varphi_1 \varphi_2$ of the restrictions $\varphi_i \upharpoonright \mathbf{A}_i$, where a universal product is a state on the *free product* (with identification of units) $\mathbf{A}_1 * \mathbf{A}_2$ (i.e., the *coproduct* of \mathbf{A}_1 and \mathbf{A}_2 in the category of unital $*$ -algebras) fulfilling conditions like associativity and functoriality (i.e., the construction *commutes* with unital $*$ -homomorphisms). The conditions are motivated by the fact that, when interpreted classically (i.e., in the context of commutative unital $*$ -algebras) there is only one such universal product, namely, the tensor product of φ_1 and φ_2 .

In the non-commutative context, besides the tensor product (corresponding to *tensor independence*), we have the *free product* of states which corresponds to *free independence* introduced by Voiculescu [Voi87]. Speicher [Spe97] has shown that under stronger (from the combinatorial point of view very natural) assumptions there are only those two universal products. In [BGS99] Ben Ghorbal and Schürmann show how the original set-up from [Sch95] can be reduced to [Spe97].

Allowing for non-unital $*$ -algebras, there is a third universal product, namely, the *boolean product* introduced by von Waldenfels [Wal73] which

corresponds to *boolean independence*. (Actually, there is a whole family of such products labelled by a scaling parameter; see [BGS99]. We consider only the simplest choice of this parameter.) The boolean product is in some sense the simplest possible product, as it just factorizes on alternating monomials, i.e., the boolean product sends a monomial $a_1 a_2 \dots$ where two neighbours are from different algebras just to the product $\varphi_j(a_1) \varphi_k(a_2) \dots$ where a_i must be evaluated in the appropriate state, i.e., $j=1$ for $a_1 \in A_1$ and $j=2$ for $a_1 \in A_2$, and so on.

Each type of independence has its own type of Fock space which is suggested by the GNS-construction for the respective product states; see [Sch95] for details. For tensor indepenence this is the symmetric Fock space. (This is mirrored by the well-known factorization $\Gamma(H_1 \oplus H_2) = \Gamma(H_1) \otimes \Gamma(H_2)$.) For free independence this is the full Fock space. (This is mirrored by the fact that $\mathcal{F}(H_1 \oplus H_2)$ is the *free product* of $\mathcal{F}(H_1)$ and $\mathcal{F}(H_2)$ with their respective vaccua as reference vector.) The *boolean Fock space* over H is just $\mathcal{F}_b(H) = \mathbb{C}\Omega \oplus H$. (Here the composition law is just the direct sum of the one-particle sectors. We may view this a direct sum of $\mathcal{F}_b(H_i) = \mathbb{C}\Omega_i \oplus H_i$ ($i=1, 2$) with *identification of the reference vectors* Ω_i .)

The primary goal of this section is to discover a calculus on the boolean Fock space. Similar to the symmetric and the full Fock space the solution of a differential equation like (11.1) should be a process with stationary Boolean independent multiplicative increments; cf. Section 11. The way we find this calculus is to assign to a Hilbert space H (i.e., a Hilbert \mathbb{C} -module) a suitable $\tilde{\mathbb{C}}$ - $\tilde{\mathbb{C}}$ -module structure (where $\tilde{\mathbb{C}}$ denotes the *unitization* of the unital C^* -algebra \mathbb{C}). Then the full Fock module over this $\tilde{\mathbb{C}}$ -module turns out to be (up to one vector) the boolean Fock space. However, for a couple of reasons we find it convenient to start from the beginning with amalgamated versions. First, the C^* -algebra $\tilde{\mathbb{C}}$ is a source of continuous confusion of the several different copies of \mathbb{C} which appear in this context. Second, the examples without amalgamation are rather poor and can easily be computed by hand. Last but not least, we classify the uniformly continuous contractive, but, not necessarily conservative CP-semigroups T on a unital C^* -algebra B which may be dilated with the help of an almaga-mated boolean calculus, as those which are of the form $T_t(b) = e^{t^*} b e^t$ for suitable $j \in B$.

Like everywhere in these notes B is a unital C^* -algebra. Following Voiculescu [Voi95], B is typically the range of a certain conditional expectation $\varphi: A \rightarrow B$ preserving the unit 1 of A . This generalizes the notion of a state on A which takes values in the unital C^* -algebra \mathbb{C} .

The *unitization* of B is the C^* -algebra $\tilde{B} = B \oplus \tilde{1}\mathbb{C}$ with the new unit $\tilde{1}$. As C^* -algebra \tilde{B} isomorphic to $B \oplus \mathbb{C}$ where here the direct sum is that of C^* -algebras.

Let E be a Hilbert \mathbf{B} -module. As \mathbf{B} is an ideal in $\tilde{\mathbf{B}}$, we are free to consider E also as a Hilbert $\tilde{\mathbf{B}}$ -module. Also $\mathcal{B}^a(E)$ is the same no matter whether we consider E as a \mathbf{B} - or a $\tilde{\mathbf{B}}$ -module. By setting $bx = 0$ for $b \in \mathbf{B}$ and $\tilde{\mathbf{1}}x = x$ ($x \in E$) we define left multiplication by elements of $\tilde{\mathbf{B}}$ on E . Thus, E is a Hilbert $\tilde{\mathbf{B}}$ - $\tilde{\mathbf{B}}$ -module.

14.1. PROPOSITION. $\mathcal{F}(E) = \tilde{\mathbf{B}} \oplus E$.

Proof. Let $x, y \in E$. Then $x \odot y = x\mathbf{1} \odot y = x \odot \mathbf{1}y = 0$, whence, $E \odot E = \{0\}$. ■

On $\mathcal{F}(E)$ we may define a projection $q: x \mapsto x\mathbf{1}$. The range of q is the Hilbert \mathbf{B} -module $\mathcal{F}_b(E) = \mathbf{B} \oplus E$. Its orthogonal complement is the one-dimensional subspace spanned by the element $\tilde{\mathbf{1}} - \mathbf{1}$ of $\tilde{\mathbf{B}}$. We may think of $\mathcal{F}_b(E)$ as the *boolean Fock module* over E . (This may be justified by giving a formal definition of *boolean independence with amalgamation over \mathbf{B}* paralleling that of Voiculescu [Voi95] for free independence and that of [Ske96, Ske99] for tensor independence. We do not follow this idea here.)

14.2. PROPOSITION. q is a central projection in $\mathcal{B}^a(\mathcal{F}(E))$. Moreover, the ideal $q\mathcal{B}^a(\mathcal{F}(E))$ in $\mathcal{B}^a(\mathcal{F}(E))$ is isomorphic to $\mathcal{B}^a(\mathcal{F}_b(E))$ and has codimension 1. Consequently, $\mathcal{B}^a(\mathcal{F}(E)) \cong \mathcal{B}^a(\mathcal{F}_b(E)) \sim$. In other words,

$$\mathcal{B}^a(\mathcal{F}(E)) = \mathbb{C}(\mathbf{1}_{\mathcal{F}} - q) \oplus \mathcal{B}^a(\mathcal{F}_b(E)) = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathcal{B}^a(\mathcal{F}_b(E)) \end{pmatrix} \text{ acting on}$$

$$\mathcal{F}(E) = \begin{pmatrix} \mathbb{C}(\tilde{\mathbf{1}} - \mathbf{1}) \\ \mathcal{F}_b(E) \end{pmatrix},$$

where $\mathbf{1}_{\mathcal{F}}$ denotes the unit in $\mathcal{B}^a(\mathcal{F}(E))$.

Proof. Let $a \in \mathcal{B}^a(\mathcal{F}(E))$. Then $qax = (ax)\mathbf{1} = a(x\mathbf{1}) = aqx$. From this the remaining statements are obvious. ■

As $\mathcal{F}_b(E) = \begin{pmatrix} \mathbf{B} \\ E \end{pmatrix}$, we may decompose also $\mathcal{B}^a(\mathcal{F}_b(E)) = \begin{pmatrix} \mathbf{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}$. (Notice that a mapping $\Phi: E \rightarrow \mathbf{B}$ is in $\mathcal{B}^a(E, \mathbf{B})$, if and only if it is of the form $\Phi(x) = \langle y, x \rangle$ where $y = \Phi^*(\mathbf{1}) \in E$.) We find

$$\mathcal{B}^a(\mathcal{F}(E)) = \begin{pmatrix} \mathbb{C} & 0 & 0 \\ 0 & \mathbf{B} & E^* \\ 0 & E & \mathcal{B}^a(E) \end{pmatrix} \quad \text{acting on} \quad \mathcal{F}(E) = \begin{pmatrix} \mathbb{C}(\tilde{\mathbf{1}} - \mathbf{1}) \\ \mathbf{B} \\ E \end{pmatrix}. \quad (14.1)$$

Now let E and F be Hilbert \mathbf{B} -modules both equipped with the Hilbert $\tilde{\mathbf{B}}$ - $\tilde{\mathbf{B}}$ -module structure as described above. Then

$$\begin{aligned}\mathcal{F}(E \oplus F) &= \mathcal{F}(E) \odot (\tilde{\mathbf{B}} \oplus F \odot \mathcal{F}(E \oplus F)) \\ &= (\tilde{\mathbf{B}} \oplus E) \odot (\tilde{\mathbf{B}} \oplus F) = \mathcal{F}(E) \odot \mathcal{F}(F)\end{aligned}$$

(which, of course, equals $\tilde{\mathbf{B}} \oplus E \oplus F$ as $E \odot F = \{0\}$).

14.3. PROPOSITION. $q \in \mathcal{B}^a(\mathcal{F}(E \oplus F))$ is not adapted to E .

Proof. Let $X = (1, 0, 0) \in \mathcal{F}(E)$ and $Y = (1, 0, y) \in \mathcal{F}(F)$ ($y \neq 0$) as in (14.1). Then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}$$

in $\mathcal{F}(E \oplus F)$. Applying q to this vector, we obtain $(0, 0, y)$. However, as

$$\begin{pmatrix} \beta \\ b \\ x \end{pmatrix} \odot \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} \beta \\ 0 \\ \beta y + x \end{pmatrix},$$

there is no vector $X' \in \mathcal{F}(E)$ such that $X' \odot Y = q(X \odot Y)$. *A fortiori* there is no operator a on $\mathcal{F}(E)$ such that $q = a \odot \text{id}$. ■

This property makes the definition of adaptedness to E of operators on $\mathcal{F}_b(E \oplus F)$ a little bit delicate. If q was adapted, we would just say that an operator on $\mathcal{F}_b(E \oplus F)$ is adapted, if it can be written as qa for some operator a adapted to E in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$. Here we must be more careful. We say $a \in \mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ is *adapted to E* , if it is adapted to E in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$. In other words, we consider the intersection of the algebra of adapted operators on $\mathcal{F}(E \oplus F)$ with $q\mathcal{B}^a(\mathcal{F}(E \oplus F)) = \mathcal{B}^a(\mathcal{F}_b(E \oplus F))$.

Proposition 14.3 means that the unit q of $\mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ is not adapted. Fortunately, all the operators $p(T)$ ($T \in \mathcal{B}^a(E)$), $\ell^*(x)$ ($x \in E$) and, consequently also $\ell(x)$ remain unchanged, if we multiply them by q . (In other words, they belong to $\mathcal{B}^a(\mathcal{F}_b(E \oplus F))$.) So it is very well possible to find a calculus on $\mathcal{F}_b(L^2(\mathbb{R}, E))$ by restricting the calculus on $\mathcal{F}(L^2(\mathbb{R}, E))$ to adapted processes with values in $\mathcal{B}^a(\mathcal{F}_b(L^2(\mathbb{R}, E)))$. We may, however, not hope to find unitary solutions of differential equations.

We close the discussion of $\mathcal{F}_b(E \oplus F)$ by writing down the adapted operators.

14.4. PROPOSITION. *An operator a on $\mathcal{F}_b(E \oplus F) = \mathbf{B} \oplus E \oplus F$ is adapted to E , if and only if it is the extension by 0 to $0 \oplus 0 \oplus F$ of an operator a' on $\mathcal{F}_b(E)$.*

Proof. One easily checks that the subspace F of $\mathcal{F}_b(E \oplus F) \subset \mathcal{F}(E \oplus F)$ consists of all elements of the form $(\tilde{\mathbf{1}} - \mathbf{1}) \odot y$. Then for $a' \in \mathcal{F}_b(E)$ we have $(a' \odot \text{id})((\tilde{\mathbf{1}} - \mathbf{1}) \odot y) = 0$, because $a'(\tilde{\mathbf{1}} - \mathbf{1}) = 0$. Conversely, if $ay = 0$ for all $y \in F$, then a restricts to an operator a' on $\mathcal{F}_b(E)$ such that $a' \odot \text{id} = a$, as before. ■

14.5. THEOREM. *Let $q_E \in \mathcal{B}^a(\mathcal{F}(E \oplus F))$ denote the projection onto $\mathcal{F}_b(E)$. An operator a on $\mathcal{F}(E \oplus F) = \tilde{\mathbf{B}} \oplus E \oplus F$ is adapted to E , if and only if it is the sum of an operator $a' \in \mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ adapted to E and a multiple of $\mathbf{1}_{\mathcal{F}} - q_E$.*

Proof. The operator $\mathbf{1}_{\mathcal{F}} - q_E$ is the difference of $\mathbf{1}_{\mathcal{F}}$ which is adapted and q_E which by Proposition 14.4 is also adapted. Therefore, operators of the stated form, indeed, are adapted.

Conversely, let

$$a = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & b & z^* \\ 0 & z' & a' \end{pmatrix}$$

be adapted. The part of a corresponding to $\alpha = 0$, $z \in E$, and $a' \in \mathcal{B}^a(E)$ is adapted. Thus, we may subtract this part and assume, henceforth, that $b = 0$, $z \in F$, and $a'E = \{0\}$. (Actually, we may only assume that $a'E \subset F$. However, by adaptedness of a it is impossible that a' maps an element of E to a non-zero element of F .) If we apply this modified operator a to $(0, 0, y)$, we obtain $(0, \langle z, y \rangle, a'y)$. On the other hand, as a is adapted, it must be possible to write this vector as $X \odot (0, 0, y)$. Like in the proof of Proposition 14.3 we find that the middle component of such a vector is 0. Therefore, $z = 0$. The same argument applied to a^* yields $z' = 0$. Taking this into account, a has been reduced to $\alpha(\mathbf{1}_{\mathcal{F}} - q) + a'$. We find $a(0, 0, y) = (0, 0, a'y)$. On the other hand, recall that $(0, 0, y) = (1, 0, 0) \odot (0, 0, y)$ (where we consider $(0, 0, y)$ both as element of $\mathcal{F}(E \oplus F)$ and as element of $\mathcal{F}(F)$). Clearly, an adapted operator sends such a vector to a multiple of itself. Thus, a' is a multiple of the projection $q - q_E$ onto F . As we already remarked, $\mathbf{1}_{\mathcal{F}} - q_E = (\mathbf{1}_{\mathcal{F}} - q) + (q - q_E)$ is adapted, but $\mathbf{1}_{\mathcal{F}} - q$ is not. So the only possibility for this multiple of $q - q_E$ is $\alpha(q - q_E)$. ■

14.6. COROLLARY. *Let $u \in \mathcal{B}^a(\mathcal{F}(E \oplus F))$ be a unitary adapted to E with $\alpha \in \mathbb{C}$ in its upper left corner (so that $|\alpha| = 1$). Then the unitary $qu \in \mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ is the sum of the operator $qu - \alpha(\mathbf{1} - q_E) \in \mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ adapted to E and*

the operator $\alpha(\mathbf{1} - q_E)$ adapted to F . In other words, u is the direct sum of a unitary $q_E u$ on $\mathcal{F}_b(E)$ and a unitary $\alpha(\mathbf{1}_{\mathcal{F}} - q_E)$ on $\mathbb{C}(\tilde{\mathbf{1}} - \mathbf{1}) \oplus F$, both being adapted to E .

Now we concentrate on $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}, E))$. We remark that it does not matter, whether we first construct $L^2(\mathbb{R}, E)$ for the Hilbert \mathbf{B} -module E and then turn it into a Hilbert $\tilde{\mathbf{B}}$ - $\tilde{\mathbf{B}}$ -module, or conversely. We use also the other notations as introduced in Definition 6.1. Additionally, we introduce the projections q_t onto the boolean Fock module until time t and we set $q = q_\infty$.

We have at hand all our results until Section 12. (Of course, E contains not one non-zero element commuting with any non-zero element of $\tilde{\mathbf{B}}$. Thus, E is extremely uncentered.) Additionally, as there are only the vacuum and the one-particle sector, we do not really need any notion from Section 2, although it nevertheless leads to more compact estimates, if we continue using the ℓ_1 -norm.

The truncated structure of our Fock module or, what is the same, the trivial action of \mathbf{B} on E reduces the possibilities for integrals. In a creation integral $\int F d\ell^*(x) G$ only the component of F along $\mathbf{1}_{\mathcal{F}} - q$ contributes. Absorbing the numerical time dependence of the multiple of $\mathbf{1}_{\mathcal{F}} - q$ into G , we may replace F by $\mathbf{1}_{\mathcal{F}}$. The opposite statement is true for annihilation integrals. Particularly boring are conservation integrals where only integrals of the form $\int_{\tau}^{\mathcal{T}} f(t) dp_t(T)$ with a numerical function $f \in \mathcal{C}(\mathbb{R})$ survive. Considering f as multiplication operator on E_∞ , we just obtain $p(fT\chi_{[\tau, \mathcal{T}]}) = \int_{\tau}^{\mathcal{T}} dp_t(fT)$. This means that in all non-zero places of the Ito table the processes G and F' , which are “sandwiched” between the differentials, disappear.

On the remaining sides of the integrators we may insert the *vacuum projection* $|\omega\rangle\langle\omega|$ without changing the value of the integral. Thus, we have $\int d\ell^*(x) G = \int d\ell^*(x) \hat{\ell}(G^*\omega)$, $\int F d\ell(x) = \int \hat{\ell}^*(F\omega) d\ell(x)$ and $\int F d\mu^{x, y} G = \int \hat{\ell}^*(F\omega) d\mu^{x, y} \hat{\ell}(G^*\omega)$.

Let U be a left adapted cocycle obtained as solution of the differential equation as in Theorem 11.2. We write this in integral form and obtain

$$U_t = \mathbf{1}_{\mathcal{F}} + \int_0^t U_s(dp_s(t) + d\ell_s^*(\zeta) + d\ell_s(\zeta) + J ds). \quad (14.2)$$

The following projection procedure can be done in more steps what yields interesting intermediate results, in particular, in the context of product systems of Hilbert modules; see [BS99]. (We already established $\mathcal{F}(E) \odot \mathcal{F}(F) = \mathcal{F}(E \oplus F)$ for modules with the trivial left action of \mathbf{B} . On the other hand, this is reconfirmed by the result in [BS99] that the *time ordered Fock modules* form a product system and that for our modules time

ordered and full Fock module coincide, because the difference appears first in the two-particle sector.)

Let us multiply (14.2) by $\mathbf{1} \in \mathbf{B} \subset \mathcal{B}^a(\mathcal{F})$ from both sides. (Due to the trivial action of \mathbf{B} this corresponds more or less to the vacuum conditional expectation on the boolean Fock module. Thus, the result may be interpreted as an element of \mathbf{B} .) Then all integrals except the time integral are cancelled. We obtain

$$\mathbf{1}U_t\mathbf{1} = \mathbf{1} + \int_0^t \mathbf{1}U_s\mathbf{1}J ds.$$

In other words, setting $b_t = \mathbf{1}U_t\mathbf{1} \in \mathbf{B}$, we find $b_t = e^{\eta}$. This means that we obtain a CP-semigroup T of the very special form

$$T_t(b) = \langle \omega, U_t b U_t^{-1} \omega \rangle = b_t b b_t^*.$$

(Notice that we did not even require U_t to be unitary.)

Conversely, let E be a Hilbert \mathbf{B} -module equipped with the usual Hilbert $\tilde{\mathbf{B}}$ - $\tilde{\mathbf{B}}$ -module structure. If $T_t(b) = \langle \xi_t, b \xi_t \rangle$ ($\xi_t \in \mathcal{F}$) defines a semigroup on \mathbf{B} , then for the components $b_t^* = \mathbf{1} \xi_t \in \mathbf{B}$ of ξ_t ; we necessarily have $b_s b_t b_t^* b_s^* = b_{s+t} b b_{s+t}^*$ for all $b \in \mathbf{B}$; $s, t \in \mathbb{R}^+$. Of course, this does not necessarily mean that $b_s b_t = b_{s+t}$. One can, however, show with the help of product systems (where the ξ_t form a so-called *unit*; see [BS99]) that the b_t can be chosen accordingly. Together with the assumption that T has a bounded generator we arrive at the same conclusion.

And yet another way to look at it is to start with a uniformly continuous conservative CP-semigroup \tilde{T} on $\tilde{\mathbf{B}}$. Then the GNS-module E of the completely positive part L^0 of the generator L has the desired $\tilde{\mathbf{B}}$ - $\tilde{\mathbf{B}}$ -module structure, if and only if $L^0(b) = 0$ for all $b \in \mathbf{B}$. In this case, the restriction of L to \mathbf{B} has the form $L(b) = Jb + bJ^*$ where $\operatorname{Re} J = -\frac{1}{2} \langle \xi, \xi \rangle \in \mathbf{B}$ and $\operatorname{Im} J = \mathbf{1}h \in \mathbf{B}$. Once again, one verifies directly by differentiation that $T_t(b) = e^{\eta} b e^{\eta^*}$ has this generator. Additionally, we see that \tilde{T} is the unital extension of a contractive uniformly continuous CP-semigroup T from \mathbf{B} to $\tilde{\mathbf{B}}$.

Contractive CP-semigroups on \mathbb{C} have the form $T_t(z) = e^{-tc}z$ ($c \geq 0$). We have discussed the corresponding truncated Fock $\tilde{\mathbb{C}}$ - $\tilde{\mathbb{C}}$ -module $L^2(\mathbb{R}^+) \oplus \tilde{\mathbf{1}}$ at length in [BS99, Sect. 8].

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